\[
\det(A) = \sum_{\sigma \in S_n} \text{Sign}(\sigma) a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}
\]

**Theorem** Suppose \( B \) is formed from \( A \) by multiplying row \( k \) by the scalar \( \alpha \). Then \( \det(B) = \alpha \det(A) \)

**Proof**
\[
\det(B) = \sum_{\sigma \in S_n} \text{sign}(\sigma) b_{\sigma(1)} b_{\sigma(2)} \cdots b_{\sigma(n)}
\]
\[
= \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{\sigma(1)} \cdots (\alpha a_{\sigma(k)}) \cdots a_{\sigma(n)}
\]
\[
= \alpha \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{\sigma(1)} \cdots a_{\sigma(n)} = \alpha \det(A)
\]
Corollary: If $A$ has a row of all zeros, then $\det(A) = 0$.

Proof: Let $B$ be the matrix $A$, except the row of zeros is multiplied by zero. Note: $B = A$

$$\det(A) = \det(B) = 0 = \det(A)$$

Theorem: Suppose $B$ is the matrix obtained from $A$ by swapping rows $i$ and $j$. Then $\det(B) = -\det(A)$

Proof: Let $\pi$ be permutation that swaps $i$ and $j$, fixes everything else.

$$\pi = 1 \ 2 \ \cdots \ j \ \cdots \ i \ \cdots \ n \ n \ \cdots \ i < j$$

$$\det B = \sum_{\sigma \in S_n} \text{sign}(\sigma) \ b_{\sigma(i)} b_{\sigma(1)} \cdots b_{\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \text{sign}(\sigma \pi) \ b_{\sigma(\pi(i))} b_{\sigma(\pi(1))} \cdots b_{\sigma(\pi(n))}$$

$$= -\sum_{\sigma \in S_n} \text{sign}(\sigma) \ b_{\sigma(1)} b_{\sigma(2)} \cdots b_{\sigma(n)}$$
$$= \sum_{\sigma \in S_n} (-1)^{\text{sign}(\sigma)} b_{i \sigma(i)} \cdots b_{j \sigma(j)} \cdots b_{n \sigma(n)}$$

$$= \sum_{\sigma \in S_n} (-1)^{\text{sign}(\sigma)} a_{i \sigma(i)} \cdots a_{j \sigma(j)} \cdots a_{n \sigma(n)}$$

$$= (-1)^{\text{sign}(\sigma)} a_{i \sigma(i)} \cdots a_{i \sigma(i)} \cdots a_{j \sigma(j)} \cdots a_{n \sigma(n)}$$

$$= (-1)^{\text{det}(A)}.$$ 

**Corollary**  Suppose $A$ has two identical rows, then $\det(A) = 0$.

**Proof**  Let $B$ be the matrix formed from $A$ by swapping the two rows.  Then $A = B$ and $\det(B) = -\det(A) \Rightarrow 2\det(A) = 0 \Rightarrow \det(A) = 0.$
Theorem: Form matrix $B$ by adding a multiple of row $i$ to row $j$ (from $A$).

Then: \( \det(A) = \det(B) \).

Proof:
\[
\det B = \sum_{\sigma \in S_n} \text{sign}(\sigma) \cdot b_{\sigma(i)} \ldots b_{\sigma(n)}
\]

\[
= \sum_{\sigma \in S_n} \text{sign}(\sigma) \cdot a_{\sigma(i)} \ldots (\alpha a_{\sigma(i)} + a_{\sigma(j)}) \ldots a_{\sigma(n)}
\]

\[
= \sum_{\sigma \in S_n} \text{sign}(\sigma) \cdot a_{\sigma(i)} \ldots a_{\sigma(j)} \ldots a_{\sigma(n)}
+ \sum_{\sigma \in S_n} \text{sign}(\sigma) \cdot a_{\sigma(i)} \ldots a_{\sigma(j)} \ldots a_{\sigma(n)}
\]

\[
= 0 + \det(A)
\]

determinant of a matrix where rows $i$ & $j$ are identical.
Axiomatic Definition of Determinant

Define a function of a matrix to the complex numbers, using the rows of the matrix (as vectors).

\[ D : \mathbb{M}_{n \times n} \rightarrow \mathbb{C} \quad D (r_1, r_2, \ldots, r_n) \]

**Require**

1. \[ D (r_1, \ldots, \alpha r_k, \ldots, r_n) = \alpha D(r_1, \ldots, r_n) \]
2. \[ D(r_1, \ldots, r_k + r_s, \ldots, r_n) = D(r_1, \ldots, r_k, \ldots, r_n) + D(r_1, \ldots, r_s, \ldots, r_n) \]
3. \[ D(r_1, \ldots, r_k, \ldots, r_k, \ldots, r_n) = 0 \]
4. \[ D(e_1, e_2, \ldots, e_n) = 1 \]

**D????**