Axiomatic Definition

\[ D: \mathbb{M}_n \rightarrow \mathbb{C} \]

1. \[ D(\alpha r, \cdot) = \alpha D( r, \cdot) \]
2. \[ D( r, r + r') = D( r, \cdot) + D( r', \cdot) \]
3. \[ D( r, r, \cdot) = 0 \]
4. \[ D(e_1, \ldots, e_n) = 1 \]

In fact, matrix with 0 row \[ \Rightarrow \text{determinant is zero} \]

\[ D(0, e_1, \ldots, e_n) = D(0, 0, \ldots, 0) = 0 = D(r, 0, \ldots, 0) \]

Determinants

Tue - Problem Session

Evaluations

Thu \_ Talk Preparation

Fri \_ Mon - Anna

Tristen

Tue - Hayden

Jack
Fact: Swapping rows changes the sign of the determinant.

\[
0 = D(\cdots, r_i + r_j, \cdots, r_i + r_j, \cdots) 
\]

(3)

\[
\Rightarrow D(\cdots, r_i, \cdots, r_j, \cdots) + D(\cdots, r_i, \cdots, r_i, \cdots) + D(\cdots, r_j, \cdots, r_i, \cdots) + D(\cdots, r_j, \cdots, r_j, \cdots) = 0 \quad (\text{fact})
\]

(2) Hence, for \( i \neq j \),

\[
= D(\cdots, r_i, \cdots, r_j, \cdots) + D(\cdots, r_j, \cdots, r_i, \cdots)
\]

n (negatives of each other)

Fact: Matrix with linearly dependent rows \( \Rightarrow \) determinant 0, plus i

\[
r_i = a_1 r_1 + a_2 r_2 + \cdots + a_n r_n
\]

skip \( r_i \)

D(\cdots, r_i, \cdots) = D(\cdots, a_1 r_1 + \cdots + a_n r_n, \cdots) = \sum_{k=1}^{n} a_k D(\cdots, r_k, \cdots)

matrix w/ two equal rows

\[
= \sum_{k \neq i}^{n} a_k \cdot 0 = 0
\]

(3)
Determinant of a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\text{det}(A) = \text{D}(ae_1 + be_2, ce_1 + de_2)$$

$$= \text{D}(ae_1, ce_1) + \text{D}(ae_1, de_2) + \text{D}(be_2, ce_1) + \text{D}(be_2, de_2)$$

$$= ac \text{D}(e_1, e_1) + ad \text{D}(e_1, e_2) + bc \text{D}(e_2, e_1) + bd \text{D}(e_2, e_2)$$

$$= ac \cdot 0 + ad \cdot 1 + bc (-1) + bd \cdot 0$$

$$= (3) + (4) + \text{fact}$$

$$= ad - bc$$

**Fact**: The determinant of a diagonal matrix is the product of the diagonal entries.

$$\text{D}(a_1 e_1, a_2 e_2, \ldots, a_n e_n) = a_1 \text{D}(e_1, a_2 e_2, \ldots, a_n e_n)$$

$$= \cdots = a_{11} a_{22} \ldots a_{nn} \text{D}(e_1, e_2, \ldots, e_n) = a_{11} a_{22} \cdots a_{nn} \cdot 1$$
The determinant of an upper-triangular matrix is the product of the diagonal entries.

1) A 0 entry on the diagonal $\Rightarrow$ determinant is zero

$$
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  0 & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & a_{nn}
\end{bmatrix}
$$

These vectors are linearly dependent, i.e., $n-k+1$ vectors with $n-k$ slots $\in \mathbb{C}^{n-k}$

2) Row 1, rewrite as $d + x = (a_{11}, 0, \ldots, 0) + (0, a_{12}, \ldots, a_{1n})$

$$
\text{determinant} = D(d+x, \ldots) = D(d, \ldots) + D(x, \ldots)
$$

But $D(x, \ldots) = 0$ on diagonal (02, first column zero vector)

Repeat w/ subsequent rows

\[ \text{looking more diagonal} \]
How to compute $D(A)$

1) $A$ singular $\Rightarrow$ linearly dependent rows $\Rightarrow$ $D(A) = 0$

2) $A$ nonsingular $\Rightarrow$ $A$ invertible $\Rightarrow$ $I_n = A^{-1}A$

$In = E_n \cdots E_2 E_1 A$

Elementary matrices

that "do" row operations

Track the effect of the row operations (swap, multiply row, multiply a row & add to another)

So there is only one way to compute $D(A)$. 