Math 390 Monday, April 26 Determinants

Tue - Problem Session
Evaluations

Thu
Fri] Talk Preparation

Mon - Anna
Tristan

Tue - Mayden
Jack

Axiomatic Definition

$D: M_{nn} \longrightarrow \mathbb{C}$

1) $D(\quad , \alpha \underset{\sim}{r_i}, \quad) = \alpha D(\quad , \underset{\sim}{r_i}, \quad)$

2) $D(\quad , r_i + \underset{\sim}{r}, \quad) = D(\quad , \underset{\sim}{r_i}, \ldots) + D(\quad , \underset{\sim}{r}, \quad)$

3) $D(\quad , \underset{\sim}{r_k}, \quad , r_k, \quad) = 0$

4) $D(\underbrace{\underset{\sim}{e_1}, \ldots, \underset{\sim}{e_n}}_{I_n}) = 1$

<u>Fact</u> Matrix w/ $\underset{\sim}{0}$ row $\Rightarrow$ determinant is zero

$$D(\quad , \underset{\sim}{0}, \quad) = D(\quad , 0\underset{\sim}{0}, \quad) \underset{(1)}{=} 0\, D(\quad , \underset{\sim}{0}, \quad) = 0$$

<u>Fact</u> Swapping rows changes the sign of the determinant

$$0 \overset{(3)}{=} D(\ , \mathbf{r}_i + \mathbf{r}_j, \dots, \mathbf{r}_i + \mathbf{r}_j, \dots)$$

$$\overset{\text{(2) twice FOIL}}{=} D(\ , \mathbf{r}_i, \dots \mathbf{r}_j, \dots) + \underbrace{D(\mathbf{r}_i, \dots, \mathbf{r}_i)}_{0\ (\text{fact})} + D(\mathbf{r}_j, \dots, \mathbf{r}_i) + \underbrace{D(\dots \mathbf{r}_j, \dots, \mathbf{r}_j \dots)}_{0\ (\text{fact})}$$

$$= D(\mathbf{r}_i, \dots, \mathbf{r}_j) + D(\mathbf{r}_j, \dots, \mathbf{r}_i)$$

negatives of each other

<u>Fact</u> Matrix w/ linearly dependent rows $\Rightarrow$ determinant 0

$$\mathbf{r}_i = a_1 \mathbf{r}_1 + a_2 \mathbf{r}_2 + \dots + a_n \mathbf{r}_n$$

skip $\mathbf{r}_i$

slot i

$$D(\ \mathbf{r}_i\ ) = D(\ , a_1 \mathbf{r}_1 + \dots + a_n \mathbf{r}_n, \ ) = \sum_{\substack{k=1 \\ k \neq i}}^{n} a_k \underbrace{D(\ , \mathbf{r}_k, \ )}_{\text{matrix w/ two equal rows}}$$

$$= \sum_{\substack{k \neq i \\ k=1}}^{n} a_k \cdot \underset{(3)}{0} = 0$$

Determinant of a 2x2 $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A) = D(a\underset{\sim}{e_1} + b\underset{\sim}{e_2},\ c\underset{\sim}{e_1} + d\underset{\sim}{e_2})$$

$$= D(a\underset{\sim}{e_1}, c\underset{\sim}{e_1}) + D(a\underset{\sim}{e_1}, d\underset{\sim}{e_2}) + D(b\underset{\sim}{e_2}, c\underset{\sim}{e_1}) + D(b\underset{\sim}{e_2}, d\underset{\sim}{e_2})$$

$$= ac\, D(\underset{\sim}{e_1}, \underset{\sim}{e_1}) + ad\, D(\underset{\sim}{e_1}, \underset{\sim}{e_2}) + bc\, D(\underset{\sim}{e_2}, \underset{\sim}{e_1}) + bd\, D(\underset{\sim}{e_2}, \underset{\sim}{e_2})$$

$$= \underset{(3)}{ac \cdot 0} + \underset{(4)}{ad\ 1} + \underset{(4) + \text{fact}}{bc(-1)} + bd \cdot 0$$

$$= ad - bc$$

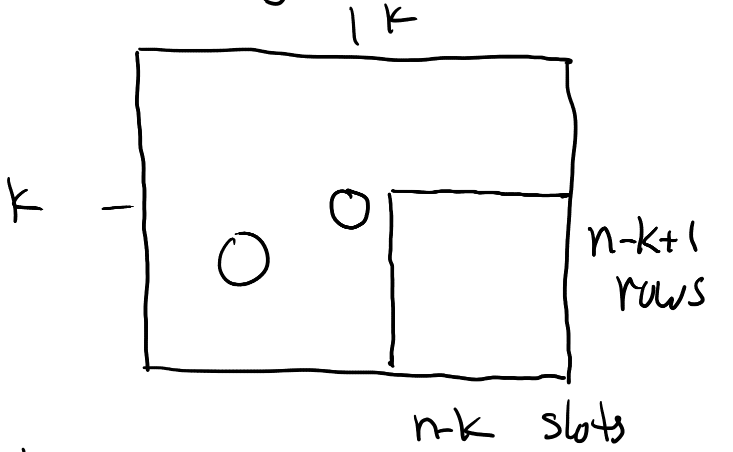
<u>Fact</u> Determinant of diagonal matrix is the product of the diagonal entries

$$D(a_{11}\underset{\sim}{e_1}, a_{22}\underset{\sim}{e_2}, \dots, a_{nn}\underset{\sim}{e_n}) = a_{11} D(\underset{\sim}{e_1}, a_{22}\underset{\sim}{e_2}, \dots, a_{nn}\underset{\sim}{e_n})$$

$$= \cdots = a_{11}a_{22}\cdots a_{nn}\, D(\underset{\sim}{e_1}, \underset{\sim}{e_2}, \dots, \underset{\sim}{e_n}) = a_{11}a_{22}\cdots a_{nn} \cdot 1$$

Fact The determinant of an upper-triangular matrix is the product of the diagonal entries.

1) A 0 entry on the diagonal $\Rightarrow$ determinant is zero



These vectors are linearly dependent

MVSLD $n-k+1$ vectors w/ $n-k$ slots $\in \mathbb{C}^{n-k}$

Linearly dependent rows $\Rightarrow$ 0 determinant

2) Row 1, rewrite as $\underset{\sim}{l} + \underset{\sim}{r} = (a_{11}, 0, \dots, 0) + (0, a_{12}, \dots, a_{1n})$

determinant $= D(\underset{\sim}{l}+\underset{\sim}{r}, \dots) = D(l, \dots) + D(\underset{\sim}{r}, \dots)$

$= D(l, \dots)$

looking more diagonal

0 on diagonal

(or first column zero vector)

repeat w/ subsequent rows

How to compute $D(A)$

1) A singular $\Rightarrow$ linearly dependent rows $\Rightarrow$ $D(A) = 0$

2) A nonsingular $\Rightarrow$ A invertible $\Rightarrow$ $I_n = \underline{A^{-1}} A$

① inverse, unique

② Accumulation of row operations

$$I_n = \underbrace{E_n \cdots E_2 E_1}_{\text{elementary matrices that "do" row operations}} A$$

Track the effect of the row operations (swap, multiply row, multiply a row & add to another)

So there is only one way to compute $D(A)$.