Using Minimum Degree to Bound Average Distance

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Abstract

We show the average distance μ of a connected graph with n vertices, e edges and minimum degree d satisfies

$$\mu \leq \frac{\left\lfloor \frac{(n+1)n(n-1)-2e}{d+1} \right\rfloor}{n\,(n-1)} \leq \frac{n+1}{d+1} - \frac{2e}{(d+1)\,n\,(n-1)}.$$

This improves conjectures of the computer program **Graffiti** [8], and He and Li [11] and the results of Kouider and Winkler [13]. The bound is sharp since complete graphs yield equality.

1 Introduction

Let $G = (V, E)$ be a connected, simple, undirected graph on n vertices. For each unordered pair of vertices $\{u, v\}$ in G, select a path of minimal length. Designate the corresponding directed paths from u to v and from v to u as, $P\,(u, v)$ and $P\,(v, u)$, respectively. The number of edges in $P\,(u, v)$ is the *distance* $\rho\,(u, v)$ from u to v and the *average distance* of G is

$$\mu = \frac{1}{n\,(n-1)} \sum_{u,v \in V} \rho\,(u, v)\,.$$

Total and average distance are not only interesting invariants of graphs in their own right but are also used for studying properties or classifying graphical systems that depend on the number of edges traversed. Recent examples include studies of computer networks [3] and the use of graphical invariants to partially classify the structure of molecules [1].

There have been a number of conjectures involving average distance in the literature including a conjecture of the computer program **G. Galatea Graffiti** [8], proven by Chung [7], that the independence number of a connected graph is at least as large as the average distance. Kouider and Winkler [13] provide an extensive list of references for other problems and results involving average distance: see [4, 5, 6, 9, 10, 12, 14, 15, 17, 18, 19].

The main result of this paper was motivated by Conjecture 62 of **Graffiti** [8]: *If G is a regular graph of degree r and order n, then the average distance of G is at most $\frac{n}{r}$.* If we denote the minimum degree of a, possibly non-regular, graph by d, then Conjecture 127 of

Graffiti [8] is $\mu \leq \frac{n}{d}$, He and Li [11] conjecture $\mu \leq \frac{n+1}{d+1}$ and Kouider and Winkler [13] prove the related result $\mu \leq \frac{n}{d+1} + 2$. In this paper we prove a more stringent version of these inequalities. Specifically,

Theorem 3.2 *Let G be a graph with n vertices, e edges, and minimum degree d. Then the average distance μ satisfies*

$$\mu \leq \frac{\left\lfloor \frac{(n+1)n(n-1)-2e}{d+1} \right\rfloor}{n\,(n-1)} \leq \frac{n+1}{d+1} - \frac{2e}{(d+1)\,n\,(n-1)}.$$

2 Lemmas

Let G be a graph with n vertices, e edges, and minimum degree d and suppose a collection of minimal paths $P(u,v)$ has been chosen. Let $N(x)$ be the set of vertices adjacent to x, let $\overline{N}(x) = N(x) \cup \{x\}$, and let $|S|$ denote the cardinality of the set S. Define $T = \{(u,v,x,y) |\, u \neq v,\ x \in P(u,v) - \{u\},\ y \in \overline{N}(x)\}$. Our first goal is to show $|T| \leq (n+1)\,n\,(n-1) - 2e$.

We consider the set T as being "constructed" by (i) selecting an arbitrary directed path $P(u,v)$, (ii) forming a 3-tuple (u,v,x) for each vertex other than u on the path, and (iii) building a 4-tuple (u,v,x,y) for each vertex y that either is adjacent to or equals x. Note:

1. Except for $u \neq v$ and $u \neq x$ the vertices u, v, x, y need not be distinct.

2. By construction, x is a vertex on a minimal path from u to v so the set $S = \{(u,v,x) |\, u \neq v,\ x \in P(u,v) - \{u\}\}$ of triples satisfying the first two conditions in the definition of T has cardinality equal to the total distance of the graph G. That is,
$$|S| = \sum_{u,v \in V} \rho(u,v).$$

3. Since d is the minimal degree of G, the third condition in the definition of T gives $(d+1)|S| \leq |T|$.

Select three, possibly identical, vertices a, b, c of G and let $T^c_{a,b} = \{x | (a,b,x,c) \in T\}$. Thus x is a vertex on path $P(a,b) - \{a\}$ and c is within distance 1 of x. The minimality of $P(a,b)$ implies $\left|T^c_{a,b}\right| \leq 3$ and it is clear that $|T| = \sum_{(a,b,c) \in V^3} \left|T^c_{a,b}\right|$.

Let V_1, V_2, V_3 be the disjoint subsets of V^3 for which elements (a,b,c) in V^3 give rise to sets $\{a,b,c\}$ with exactly one, two or three distinct vertices of V, respectively. Then

$$|T| = \sum_{(a,b,c) \in V_1} \left|T^c_{a,b}\right| + \sum_{(a,b,c) \in V_2} \left|T^c_{a,b}\right| + \sum_{(a,b,c) \in V_3} \left|T^c_{a,b}\right|.$$

In the following lemmas we investigate the terms of each of these sums.

Lemma 2.1

$$\sum_{(a,b,c)\in V_1} \left|T^c_{a,b}\right| = \sum_{a\in V} \left|T^a_{a,a}\right| = 0$$

Proof There are no paths $P(a,a)$ to use to form the 4-tuples of $|T|$. ■

Lemma 2.2

$$\sum_{(a,b,c)\in V_2} \left|T^c_{a,b}\right| = 3n(n-1) - 2e$$

Proof The three possibilities for the sets $T^c_{a,b}$ are $T^c_{a,a}$, $T^a_{a,c}$, and $T^a_{c,a}$. There are no tuples in $T^c_{a,a}$ so its contribution is 0. Clearly there are $n(n-1)$ each of sets of the form $T^a_{a,c}$ and $T^a_{c,a}$. Since

$$\left|T^a_{c,a}\right| = \begin{cases} 2 & a \text{ not adjacent to } c \\ 1 & a \text{ adjacent to } c \end{cases} \quad \text{and} \quad \left|T^a_{a,c}\right| = \begin{cases} 1 & a \text{ not adjacent to } c \\ 1 & a \text{ is adjacent to } c \end{cases}$$

and both $T^a_{c,a}$ and $T^c_{a,c}$ are of form $T^a_{c,a}$, it follows that

$$\begin{aligned} \sum_{(a,b,c)\in V_2} \left|T^c_{a,b}\right| &= \sum_{a,c\in V, a\neq c} \left|T^a_{c,a}\right| + \sum_{a,c\in V, a\neq c} \left|T^a_{a,c}\right| \\ &= 2n(n-1) - 2e + n(n-1) = 3n(n-1) - 2e. \end{aligned}$$

■

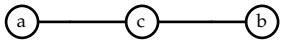
The next 3 lemmas allow us to bound the contribution of V_3 to $|T|$.

Lemma 2.3 *Suppose $a, b,$ and c are distinct vertices and that one of them, say c, is on the minimal path formed by the other two (and hence a and b are not adjacent). Then the six ordered 3-tuples corresponding to a, b, c contribute 6 to $|T|$.*

Proof We consider the three cases: c adjacent to both, c adjacent to one, and c adjacent to neither of a, b.

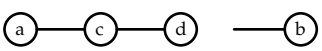
If c is adjacent to both a and b, then

$$\begin{aligned} T^c_{a,b} &= \{c,b\} \qquad T^b_{a,c} = \{c\} \qquad T^a_{b,c} = \{c\} \\ T^c_{b,a} &= \{a,c\} \qquad T^b_{c,a} = \emptyset \qquad T^a_{c,b} = \emptyset. \end{aligned}$$



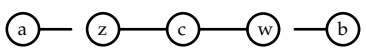
If c is adjacent to just one of a, b, say a, then

$$\begin{aligned} T^c_{a,b} &= \{c,d\} \qquad T^b_{a,c} = \emptyset \qquad T^a_{b,c} = \{c\} \\ T^c_{b,a} &= \{a,c,d\} \qquad T^b_{c,a} = \emptyset \qquad T^a_{c,b} = \emptyset. \end{aligned}$$



If c is adjacent to neither of a, b, then

$$\begin{aligned} T^c_{a,b} &= \{z, c, w\} \qquad T^b_{a,c} = \emptyset \qquad T^a_{b,c} = \emptyset \\ T^c_{b,a} &= \{z, c, w\} \qquad T^b_{c,a} = \emptyset \qquad T^a_{c,b} = \emptyset. \end{aligned}$$



Thus in all cases

$$\left|T^c_{a,b}\right| + \left|T^b_{a,c}\right| + \left|T^a_{b,c}\right| + \left|T^c_{b,a}\right| + \left|T^b_{c,a}\right| + \left|T^a_{c,b}\right| = 6.$$

■

Lemma 2.4 *Let a, b, c be distinct vertices such that no one is on the chosen minimal path joining the other two. Suppose that $\left|T^c_{a,b}\right| > 0$. Then $\rho(a,b) \geq \rho(b,c) + \left(\left|T^c_{a,b}\right| - 1\right)$. Moreover,*

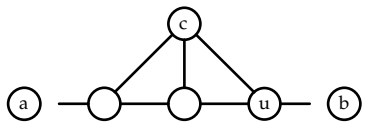
$$\begin{aligned} \text{if } \left|T^c_{a,b}\right| &= 3, \text{ then } \overline{N}(a) \cap (P(b,c) - \{b\}) = \emptyset, \\ \text{if } \left|T^c_{a,b}\right| &= 2, \text{ then } \overline{N}(a) \cap (P(b,c) - \{b\}) \subseteq \{c\}. \end{aligned}$$

Proof Let u denote the vertex in $T^c_{a,b}$ closest to b. Then minimality of the path $P(b,c)$ implies

$$\begin{aligned} \rho(b,c) &\leq 1 + \rho(u,b) \\ \rho(b,c) - 1 &\leq \rho(u,b) \leq \rho(a,b) - \left|T^c_{a,b}\right| \\ \rho(b,c) + \left|T^c_{a,b}\right| - 1 &\leq \rho(a,b). \end{aligned}$$

If $\left|T^c_{a,b}\right| = 3$, then $\rho(b,c) + 2 \leq \rho(a,b)$. Hence, a cannot be within 1 of $P(b,c)$ for, if it were, there would be a path from a through a vertex on $P(b,c)$ to b shorter than $P(a,b)$.

If $\left|T^c_{a,b}\right| = 2$, then $\rho(b,c) + 1 \leq \rho(a,b)$ and $P(a,b)$ being a shortest path again implies that a can be adjacent to just c in $P(b,c)$.



■

Lemma 2.5 *Let a, b, c be distinct vertices such that no one is on the minimal path joining the other two. Then $\left|T^c_{a,b}\right| + \left|T^a_{b,c}\right| + \left|T^b_{c,a}\right| \leq 3$, and $\left|T^c_{b,a}\right| + \left|T^b_{a,c}\right| + \left|T^a_{c,b}\right| \leq 3$.*

Proof We prove the first inequality since the second follows by a similar argument.

If none of the three terms are zero, then by Lemma 2.4

$$\begin{aligned}
\rho(a,b) &\geq \rho(b,c) + \left(\left|T^c_{a,b}\right| - 1\right) \Longrightarrow \rho(a,b) - \rho(b,c) + 1 \geq \left|T^c_{a,b}\right| \\
\rho(b,c) &\geq \rho(c,a) + \left(\left|T^a_{b,c}\right| - 1\right) \Longrightarrow \rho(b,c) - \rho(c,a) + 1 \geq \left|T^a_{b,c}\right| \\
\rho(c,a) &\geq \rho(a,b) + \left(\left|T^b_{c,a}\right| - 1\right) \Longrightarrow \rho(c,a) - \rho(a,b) + 1 \geq \left|T^b_{c,a}\right|
\end{aligned}$$

and adding the three inequalities yields $3 \geq \left|T^c_{a,b}\right| + \left|T^a_{b,c}\right| + \left|T^b_{c,a}\right|$.

If exactly one of the terms, say $\left|T^a_{b,c}\right|$, is zero, we consider three cases involving $\left|T^c_{a,b}\right|$ and $\left|T^b_{c,a}\right|$:

1. If $\left|T^b_{c,a}\right| = 3$, Lemma 2.4 implies c is not adjacent to any vertex in $P(a,b) - \{a\}$ contradicting $\left|T^c_{a,b}\right| > 0$.

2. If $\left|T^b_{c,a}\right| = 2$, then Lemma 2.4 implies that c is adjacent to at most b in $P(a,b) - \{a\}$. Thus $\left|T^c_{a,b}\right| + \left|T^b_{c,a}\right| \leq 1 + 2 = 3$.

3. If $\left|T^b_{c,a}\right| = 1$ and $\left|T^c_{a,b}\right| > 2$, then $\left|T^c_{a,b}\right| = 3$ and, as is clear from the figure in Lemma 2.4, $\rho(c,a) \leq \rho(a,b) - 1$. But $\left|T^b_{c,a}\right| = 1$ so Lemma 2.4 yields the contradiction $\rho(c,a) \geq \rho(a,b) + (1-1)$.

If exactly two of the three terms are zero and the remaining one, say $\left|T^c_{a,b}\right|$, is nonzero then $P(a,b)$ being a shortest path gives $\left|T^c_{a,b}\right| \leq 3$. ■

Lemma 2.6

$$\sum_{(a,b,c)\in V_3} \left|T^c_{a,b}\right| \leq n(n-1)(n-2).$$

Proof There are $n(n-1)(n-2)/6$ unordered triples of distinct vertices and for each such triple $\{a,b,c\}$ there are 6 ordered 3-tuples in the above sum. Thus by Lemmas 2.3 and 2.5

$$\begin{aligned}
\sum_{(a,b,c)\in V_3} \left|T^c_{a,b}\right| &= \sum_{\substack{a,b,c\in V,\\ a\neq b, b\neq c, c\neq a}} \left(\left|T^c_{a,b}\right| + \left|T^c_{b,a}\right| + \left|T^b_{a,c}\right| + \left|T^b_{c,a}\right| + \left|T^a_{b,c}\right| + \left|T^a_{c,b}\right|\right) \\
&\leq 6\binom{n}{3} = n(n-1)(n-2).
\end{aligned}$$

■

3 Main Results

Theorem 3.1 *Let G be a graph with n vertices, e edges, and minimum degree d and suppose a collection of minimal paths $P(u,v)$ has been chosen. Let $T = \{(u,v,x,y) | \, u \neq v, \; x \in P(u,v) - \{u\}, \; y \in \overline{N}(x)\}$. Then $|T| \leq (n+1)\, n\, (n-1) - 2e$.*

Proof By Lemmas 2.1, 2.2, 2.6, we have

$$\begin{aligned} |T| &= \sum_{(a,b,c)\in V_1} \left|T^c_{a,b}\right| + \sum_{(a,b,c)\in V_2} \left|T^c_{a,b}\right| + \sum_{(a,b,c)\in V_3} \left|T^c_{a,b}\right| \\ &\leq 0 + 3n\,(n-1) - 2e + n(n-1)\,(n-2) \\ &= (n+1)\, n\, (n-1) - 2e. \end{aligned}$$

■

Theorem 3.2 *Let G be a graph with n vertices, e edges, and minimum degree d. Then the average distance μ satisfies*

$$\mu \leq \frac{\left\lfloor \frac{(n+1)n(n-1)-2e}{d+1} \right\rfloor}{n\,(n-1)} \leq \frac{n+1}{d+1} - \frac{2e}{(d+1)\, n\, (n-1)}.$$

Proof Since the minimum degree of G is d, we have $(d+1)\,|S| \leq |T|$. Thus by Theorem 3.1

$$\begin{aligned} (d+1)\,|S| &\leq (n+1)\, n\, (n-1) - 2e \\ |S| &\leq \left\lfloor \frac{(n+1)\, n\, (n-1) - 2e}{d+1} \right\rfloor \\ \mu &\leq \frac{\left\lfloor \frac{(n+1)n(n-1)-2e}{d+1} \right\rfloor}{n\,(n-1)} \leq \frac{(n+1)}{d+1} - \frac{2e}{(d+1)\, n\, (n-1)}. \end{aligned}$$

■

The inequalities in Theorem 3.1 and Theorem 3.2 are sharp because complete graphs meet both bounds: for the complete graph on n vertices $\mu = 1$ and

$$\frac{(n+1)}{d+1} - \frac{2e}{(d+1)\, n\, (n-1)} = \frac{(n+1)}{n} - \frac{n(n-1)}{n^2\,(n-1)} = \frac{n+1}{n} - \frac{1}{n} = 1.$$

By considering the possible types of triples (a,b,c) it is also easy to see that both "barbell" graphs, in which two complete graphs are joined by a path, and the "path complete" graphs of Sŏltés [16], in which an isolated vertex or one end vertex of a path, is joined to at least one vertex of a complete graph, meet the bound on $|T|$. Since Sŏltés [16] proved path-complete graphs have maximum average distance for fixed n and e, maximum $|T|$ need not imply maximum average distance. It would be interesting to know if, for fixed n and e, every graph on n vertices and e edges with maximum average distance meets the bound on $|T|$.

We end by noting that Theorems 3.1 and 3.2 also provide insight into the structure of those regular graphs which have maximal average distance. Specifically, in [2], the authors

show there are regular graphs of degree r (essentially strings of complete graphs connected together so the result is regular) on n vertices where the average distance satisfies $\frac{n}{r+1} < \mu$. They also construct regular graphs for which the difference between the average distance and the bound in Theorem 3.2 is $O(1/n)$. For an example of this, let r be a multiple of 4, let u be an isolated vertex and let A and B be two copies of K_{r+1}. In each of A and B remove the edges of an $\frac{r}{4}$ matching and connect both of the end vertices of each removed edge to the vertex u. The result is a regular graph of degree r on $2r+3$ vertices with average distance, as computed in [2], and bound from Theorem 3.2

$$\mu = 2 + \frac{3r+4}{4r^2+10r+6} \leq 2 + \frac{8r+9}{4r^2+10r+6}.$$

The difference is

$$\frac{8r+9}{4r^2+10r+6} - \frac{3r+4}{4r^2+10r+6} = \frac{5(r+1)}{(2r+3)(2r+2)} = \frac{5/2}{n}.$$

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