Counting Subgraphs in Regular Graphs

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A Wine Tasting Party

Suppose you are hosting a wine-tasting party for 33 guests.

There will be 22 different wines available for tasting.

Each guest receives their own personalized "tasting card" with 8 wines listed, and so is obligated to try each of the wines listed on their card during the party.

At the end of the night we will ask for comments on "head-to-head" pairings (groups of 2) of various wines. We wish to have **exactly** 4 guests able to comment on each pair.

This is an example of a "Combinatorial Design"

Parameters are: t-(v, k, λ)

Begin with a base set of v items.

Choose a number of subsets of size k ("blocks").

We wish this collection of subsets to have the following defining property:

Every set of size t should be contained in exactly λ blocks.

A 2–(22, 8, 4) Design

Wine-tasting party is a 2–(22, 8, 4) design:
$v = 22$ wines
$k = 8$ wines on each tasting card
$t = 2$ pairs of wines for comment
$\lambda = 4$ tasting cards containing each possible pair
Requires $b = 33$ tasting cards (guests)

Nobody knows if such a design exists, or not.

Brute Force:

$\binom{22}{8} = 319770$ possible tasting cards

$\binom{319770}{33} \cong 10^{144}$ ways to select 33 of the possible cards

(10^{70} atoms in the universe?)

Check $\binom{22}{2} = 231$ pairs to see if $\lambda = 4$

Classes of Design

Steiner Triple Systems A Steiner triple system is a 2–(v, 3, 1) design. Single parameter. Studied extensively.

Regular Graph A regular graph of degree r on n vertices is a 1–(n, 2, r) design.
The base set is the $v = n$ vertices.
Every vertex (sets of size $t = 1$) is contained in exactly $\lambda = r$ blocks (edges are blocks of size $k = 2$).

Configurations Subgraphs of regular graphs are just collections of edges, i.e. collections of blocks. The analogue for a design is a "configuration," a collection of blocks. They can be classified into equivalence classes according to isomorphism and then counted.

Counting Configurations in Designs

Main Result

- There exist linear relations among the sizes of isomorphism classes of configurations in a design.
- Coefficients are functions of v and λ, and so apply equally well to any, and all, designs with common values of t and k.
- The analogue of a matching in a graph is a configuration composed of "parallel" blocks, i.e. pairwise disjoint blocks (tasting cards with no common wines).
- The opposite end of the spectrum is a "tight" configuration, which has a lot of intersection among the blocks. More precisely, every block has more than t points that are on at least two of the blocks. (For regular graphs this translates to more than 2 vertices of degree 2 or greater in each edge, i.e. no degree 1 vertices).
- Knowing the size of each isomorphism class of the tight configurations will yield the size of any of the other isomorphism classes. So the set of tight designs is known as a generating set.

3-Line Configurations, Block Size k=4

$n = 0, 1$

$\mathcal{C}_{0,0,1}$

$\mathcal{C}_{1,4,1}$

1	2	3	4

$n = 2$

$\mathcal{C}_{2,5,1}$

1	2	3	4	
1	2	3		5

$\mathcal{C}_{2,6,1}$

1	2	3	4		
1	2			5	6

$\mathcal{C}_{2,7,1}$

1	2	3	4			
1				5	6	7

$\mathcal{C}_{2,8,1}$

1	2	3	4				
				5	6	7	8

n=3

$\mathcal{C}_{3,5,1}$

1	2	3	4	
1	2	3		5
1	2		4	5

$\mathcal{C}_{3,6,1}$

1	2	3	4		
1	2	3		5	
1	2	3			6

$\mathcal{C}_{3,6,2}$

1	2	3	4		
1	2	3		5	
1	2		4		6

$\mathcal{C}_{3,6,3}$

1	2	3	4		
1	2			5	6
1		3	4	5	

$\mathcal{C}_{3,6,4}$

1	2	3	4		
1	2			5	6
		3	4	5	6

$\mathcal{C}_{3,7,1}$

1	2	3	4			
1	2	3		5		
1	2				6	7

$\mathcal{C}_{3,7,2}$

1	2	3	4			
1	2	3		5		
1			4		6	7

$\mathcal{C}_{3,7,3}$

1	2	3	4			
1	2	3		5		
			4	5	6	7

$\mathcal{C}_{3,7,4}$

1	2	3	4			
1	2			5	6	
1		3		5		7

$\mathcal{C}_{3,7,5}$

1	2	3	4			
1	2			5	6	
		3	4	5		7

$\mathcal{C}_{3,8,1}$

1	2	3	4				
1	2	3		5			
1					6	7	8

$\mathcal{C}_{3,8,2}$

1	2	3	4				
1	2			5	6		
1	2					7	8

$\mathcal{C}_{3,8,3}$

1	2	3	4				
1	2			5	6		
1		3				7	8

$\mathcal{C}_{3,8,4}$

1	2	3	4				
1	2			5	6		
		3	4			7	8

$\mathcal{C}_{3,8,5}$

1	2	3	4				
1	2			5	6		
		3		5		7	8

$\mathcal{C}_{3,8,6}$

1	2	3	4				
1				5	6	7	
	2	3	4				8

$\mathcal{C}_{3,9,1}$

1	2	3	4					
1	2			5	6			
1						7	8	9

$\mathcal{C}_{3,9,2}$

1	2	3	4					
1				5	6	7		
	2	3					8	9

$\mathcal{C}_{3,9,3}$

1	2	3	4					
1				5	6	7		
	2			5			8	9

$\mathcal{C}_{3,9,4}$

1	2	3	4					
				5	6	7	8	
1	2	3						9

$\mathcal{C}_{3,10,1}$

1	2	3	4						
1				5	6	7			
1							8	9	10

$\mathcal{C}_{3,10,2}$

1	2	3	4						
1				5	6	7			
	2						8	9	10

$\mathcal{C}_{3,10,3}$

1	2	3	4						
				5	6	7	8		
1	2							9	10

$\mathcal{C}_{3,11,1}$

1	2	3	4							
				5	6	7	8			
1								9	10	11

$\mathcal{C}_{3,12,1}$

1	2	3	4								
				5	6	7	8				
								9	10	11	12

Linear Relations for $2-(v,4,\lambda)$

$$
\begin{aligned}
\binom{v}{2}\lambda\,\mathcal{C}_{0,0,1} &= 6\,\mathcal{C}_{1,4,1} \\
6(\lambda-1)\,\mathcal{C}_{1,4,1} &= 6\,\mathcal{C}_{2,5,1}+2\,\mathcal{C}_{2,6,1} \\
4(v-4)\lambda\,\mathcal{C}_{1,4,1} &= 6\,\mathcal{C}_{2,5,1}+8\,\mathcal{C}_{2,6,1}+6\,\mathcal{C}_{2,7,1} \\
\binom{v-4}{2}\lambda\,\mathcal{C}_{1,4,1} &= 2\,\mathcal{C}_{2,6,1}+6\,\mathcal{C}_{2,7,1}+12\,\mathcal{C}_{2,8,1} \\
3(\lambda-2)\,\mathcal{C}_{2,5,1} &= 3\,\mathcal{C}_{3,5,1}+9\,\mathcal{C}_{3,6,1}+2\,\mathcal{C}_{3,6,2}+\mathcal{C}_{3,7,1} \\
6(\lambda-1)\,\mathcal{C}_{2,5,1} &= 12\,\mathcal{C}_{3,5,1}+4\,\mathcal{C}_{3,6,2}+2\,\mathcal{C}_{3,6,3}+\mathcal{C}_{3,7,2} \\
\lambda\,\mathcal{C}_{2,5,1} &= 3\,\mathcal{C}_{3,5,1}+\mathcal{C}_{3,6,3}+\mathcal{C}_{3,7,3} \\
3(v-5)\lambda\,\mathcal{C}_{2,5,1} &= 9\,\mathcal{C}_{3,6,1}+4\,\mathcal{C}_{3,6,2}+\mathcal{C}_{3,6,3}+4\,\mathcal{C}_{3,7,1}+2\,\mathcal{C}_{3,7,2}+3\,\mathcal{C}_{3,8,1} \\
(\lambda-2)\,\mathcal{C}_{2,6,1} &= \mathcal{C}_{3,6,2}+2\,\mathcal{C}_{3,7,1}+3\,\mathcal{C}_{3,8,2} \\
8(\lambda-1)\,\mathcal{C}_{2,6,1} &= 4\,\mathcal{C}_{3,6,2}+6\,\mathcal{C}_{3,6,3}+4\,\mathcal{C}_{3,7,1}+2\,\mathcal{C}_{3,7,2}+6\,\mathcal{C}_{3,7,4}+2\,\mathcal{C}_{3,8,3} \\
2(\lambda-1)\,\mathcal{C}_{2,6,1} &= 2\,\mathcal{C}_{3,6,3}+6\,\mathcal{C}_{3,6,4}+\mathcal{C}_{3,7,2}+2\,\mathcal{C}_{3,7,5}+2\,\mathcal{C}_{3,8,4} \\
4\lambda\,\mathcal{C}_{2,6,1} &= \mathcal{C}_{3,6,2}+4\,\mathcal{C}_{3,6,3}+12\,\mathcal{C}_{3,6,4}+3\,\mathcal{C}_{3,7,4}+4\,\mathcal{C}_{3,7,5}+\mathcal{C}_{3,8,5} \\
2(v-5)\lambda\,\mathcal{C}_{2,5,1} &= 2\,\mathcal{C}_{3,6,2}+2\,\mathcal{C}_{3,6,3}+2\,\mathcal{C}_{3,7,2}+4\,\mathcal{C}_{3,7,3}+3\,\mathcal{C}_{3,8,6} \\
6(\lambda-1)\,\mathcal{C}_{2,7,1} &= 3\,\mathcal{C}_{3,7,2}+4\,\mathcal{C}_{3,8,1}+2\,\mathcal{C}_{3,8,3}+2\,\mathcal{C}_{3,9,1} \\
6(\lambda-1)\,\mathcal{C}_{2,7,1} &= \mathcal{C}_{3,7,2}+6\,\mathcal{C}_{3,7,3}+2\,\mathcal{C}_{3,7,5}+2\,\mathcal{C}_{3,8,1}+2\,\mathcal{C}_{3,8,5}+3\,\mathcal{C}_{3,8,6}+\mathcal{C}_{3,9,2} \\
9\lambda\,\mathcal{C}_{2,7,1} &= 2\,\mathcal{C}_{3,7,2}+6\,\mathcal{C}_{3,7,3}+4\,\mathcal{C}_{3,7,5}+\mathcal{C}_{3,8,3}+4\,\mathcal{C}_{3,8,5}+3\,\mathcal{C}_{3,9,3} \\
\binom{v-5}{2}\lambda\,\mathcal{C}_{2,5,1} &= \mathcal{C}_{3,7,1}+\mathcal{C}_{3,7,2}+\mathcal{C}_{3,7,3}+3\,\mathcal{C}_{3,8,1}+3\,\mathcal{C}_{3,8,6}+6\,\mathcal{C}_{3,9,4} \\
(v-7)\lambda\,\mathcal{C}_{2,7,1} &= 2\,\mathcal{C}_{3,8,1}+\mathcal{C}_{3,8,3}+4\,\mathcal{C}_{3,9,1}+9\,\mathcal{C}_{3,10,1} \\
16\lambda\,\mathcal{C}_{2,8,1} &= 4\,\mathcal{C}_{3,8,4}+3\,\mathcal{C}_{3,8,6}+2\,\mathcal{C}_{3,9,2}+\mathcal{C}_{3,10,2} \\
12(\lambda-1)\,\mathcal{C}_{2,8,1} &= 2\,\mathcal{C}_{3,8,4}+3\,\mathcal{C}_{3,8,6}+\mathcal{C}_{3,9,2}+6\,\mathcal{C}_{3,9,4}+2\,\mathcal{C}_{3,10,3} \\
8(v-8)\lambda\,\mathcal{C}_{2,8,1} &= 3\,\mathcal{C}_{3,9,2}+6\,\mathcal{C}_{3,9,4}+4\,\mathcal{C}_{3,10,2}+8\,\mathcal{C}_{3,10,3}+6\,\mathcal{C}_{3,11,1} \\
\binom{v-8}{2}\lambda\,\mathcal{C}_{2,8,1} &= \mathcal{C}_{3,10,2}+2\,\mathcal{C}_{3,10,3}+6\,\mathcal{C}_{3,11,1}+18\,\mathcal{C}_{3,12,1}
\end{aligned}
$$

Solutions

$$
\begin{aligned}
\mathcal{C}_{1,4,1} &= \frac{\lambda\,(-1+v)\,v\,\mathcal{C}_{0,0,1}}{12}\\
\mathcal{C}_{2,6,1} &= \frac{(-1+\lambda)\,\lambda\,(-1+v)\,v\,\mathcal{C}_{0,0,1}}{4} - 3\,\mathcal{C}_{2,5,1}\\
\mathcal{C}_{2,7,1} &= \frac{\lambda\,(6+\lambda\,(-10+v))\,(-1+v)\,v\,\mathcal{C}_{0,0,1}}{18} + 3\,\mathcal{C}_{2,5,1}\\
\mathcal{C}_{2,8,1} &= \frac{\lambda\,(-1+v)\,v\,\left(-36+\lambda\,\left(88-17\,v+v^2\right)\right)\,\mathcal{C}_{0,0,1}}{288} - \mathcal{C}_{2,5,1}\\
\mathcal{C}_{3,7,1} &= 3\,(-2+\lambda)\,\mathcal{C}_{2,5,1} - 3\,\mathcal{C}_{3,5,1} - 9\,\mathcal{C}_{3,6,1} - 2\,\mathcal{C}_{3,6,2}\\
\mathcal{C}_{3,7,2} &= 6\,(-1+\lambda)\,\mathcal{C}_{2,5,1} - 12\,\mathcal{C}_{3,5,1} - 4\,\mathcal{C}_{3,6,2} - 2\,\mathcal{C}_{3,6,3}\\
\mathcal{C}_{3,7,3} &= \lambda\,\mathcal{C}_{2,5,1} - 3\,\mathcal{C}_{3,5,1} - \mathcal{C}_{3,6,3}\\
\mathcal{C}_{3,8,1} &= (12+\lambda\,(-13+v))\,\mathcal{C}_{2,5,1} + 12\,\mathcal{C}_{3,5,1} + 9\,\mathcal{C}_{3,6,1} + 4\,\mathcal{C}_{3,6,2} + \mathcal{C}_{3,6,3}\\
\mathcal{C}_{3,8,2} &= \frac{\lambda\,\left(2-3\,\lambda+\lambda^2\right)\,(-1+v)\,v\,\mathcal{C}_{0,0,1}}{12} + (6-3\,\lambda)\,\mathcal{C}_{2,5,1} + 2\,\mathcal{C}_{3,5,1} + 6\,\mathcal{C}_{3,6,1} + \mathcal{C}_{3,6,2}\\
\mathcal{C}_{3,8,3} &= (-1+\lambda)^2\,\lambda\,(-1+v)\,v\,\mathcal{C}_{0,0,1} + (30-24\,\lambda)\,\mathcal{C}_{2,5,1} + 18\,\mathcal{C}_{3,5,1} + 18\,\mathcal{C}_{3,6,1} + 6\,\mathcal{C}_{3,6,2} - \mathcal{C}_{3,6,3} - 3\,\mathcal{C}_{3,7,4}\\
\mathcal{C}_{3,8,4} &= \frac{(-1+\lambda)^2\,\lambda\,(-1+v)\,v\,\mathcal{C}_{0,0,1}}{4} + (6-6\,\lambda)\,\mathcal{C}_{2,5,1} + 6\,\mathcal{C}_{3,5,1} + 2\,\mathcal{C}_{3,6,2} - 3\,\mathcal{C}_{3,6,4} - \mathcal{C}_{3,7,5}\\
\mathcal{C}_{3,8,5} &= (-1+\lambda)\,\lambda^2\,(-1+v)\,v\,\mathcal{C}_{0,0,1} - 12\,\lambda\,\mathcal{C}_{2,5,1} - \mathcal{C}_{3,6,2} - 4\,\mathcal{C}_{3,6,3} - 12\,\mathcal{C}_{3,6,4} - 3\,\mathcal{C}_{3,7,4} - 4\,\mathcal{C}_{3,7,5}\\
\mathcal{C}_{3,8,6} &= \frac{2\,(6+\lambda\,(-13+v))\,\mathcal{C}_{2,5,1}}{3} + 12\,\mathcal{C}_{3,5,1} + 2\,\mathcal{C}_{3,6,2} + 2\,\mathcal{C}_{3,6,3}\\
\mathcal{C}_{3,9,1} &= \frac{(-1+\lambda)\,\lambda\,(12+\lambda\,(-16+v))\,(-1+v)\,v\,\mathcal{C}_{0,0,1}}{6} + (-54-2\,\lambda\,(-25+v))\,\mathcal{C}_{2,5,1} - 24\,\mathcal{C}_{3,5,1} - 36\,\mathcal{C}_{3,6,1} - 8\,\mathcal{C}_{3,6,2} +\\
&\quad 2\,\mathcal{C}_{3,6,3} + 3\,\mathcal{C}_{3,7,4}\\
\mathcal{C}_{3,9,2} &= \frac{(-1+\lambda)\,\lambda\,(6+\lambda\,(-16+v))\,(-1+v)\,v\,\mathcal{C}_{0,0,1}}{3} + (-48+\lambda\,(82-4\,v))\,\mathcal{C}_{2,5,1} - 30\,\mathcal{C}_{3,5,1} - 18\,\mathcal{C}_{3,6,1} - 8\,\mathcal{C}_{3,6,2} +\\
&\quad 8\,\mathcal{C}_{3,6,3} + 24\,\mathcal{C}_{3,6,4} + 6\,\mathcal{C}_{3,7,4} + 6\,\mathcal{C}_{3,7,5}\\
\mathcal{C}_{3,9,3} &= \frac{\lambda\,\left(-2+18\,\lambda+\lambda^2\,(-20+v)\right)\,(-1+v)\,v\,\mathcal{C}_{0,0,1}}{6} + (-6+27\,\lambda)\,\mathcal{C}_{2,5,1} + 8\,\mathcal{C}_{3,5,1} - 6\,\mathcal{C}_{3,6,1} + 2\,\mathcal{C}_{3,6,2} +\\
&\quad 9\,\mathcal{C}_{3,6,3} + 16\,\mathcal{C}_{3,6,4} + 5\,\mathcal{C}_{3,7,4} + 4\,\mathcal{C}_{3,7,5}\\
\mathcal{C}_{3,9,4} &= \left(-6+\frac{\lambda\,\left(140-21\,v+v^2\right)}{12}\right)\,\mathcal{C}_{2,5,1} - 9\,\mathcal{C}_{3,5,1} - 3\,\mathcal{C}_{3,6,1} - 2\,\mathcal{C}_{3,6,2} - \mathcal{C}_{3,6,3}\\
\mathcal{C}_{3,10,1} &= \frac{\lambda\,(-1+v)\,v\,\left(126+18\,\lambda\,(-19+v)+\lambda^2\,\left(244-29\,v+v^2\right)\right)\,\mathcal{C}_{0,0,1}}{162} + (18+\lambda\,(-19+v))\,\mathcal{C}_{2,5,1} +\\
&\quad 6\,\mathcal{C}_{3,5,1} + 12\,\mathcal{C}_{3,6,1} + 2\,\mathcal{C}_{3,6,2} - \mathcal{C}_{3,6,3} - \mathcal{C}_{3,7,4}\\
\mathcal{C}_{3,10,2} &= \frac{\lambda\,(-1+v)\,v\,\left(54+12\,\lambda\,(-22+v)+\lambda^2\,\left(262-29\,v+v^2\right)\right)\,\mathcal{C}_{0,0,1}}{18} + (60+\lambda\,(-130+6\,v))\,\mathcal{C}_{2,5,1} +\\
&\quad 36\,\mathcal{C}_{3,6,1} + 2\,\mathcal{C}_{3,6,2} - 22\,\mathcal{C}_{3,6,3} - 36\,\mathcal{C}_{3,6,4} - 12\,\mathcal{C}_{3,7,4} - 8\,\mathcal{C}_{3,7,5}\\
\mathcal{C}_{3,10,3} &= \frac{\lambda\,(-1+v)\,v\,\left(72+\lambda^2\,\left(204-25\,v+v^2\right)-\lambda\,\left(276-25\,v+v^2\right)\right)\,\mathcal{C}_{0,0,1}}{48} + \left(36-\frac{\lambda\,\left(252-25\,v+v^2\right)}{4}\right)\,\mathcal{C}_{2,5,1} +\\
&\quad 18\,\mathcal{C}_{3,5,1} + 18\,\mathcal{C}_{3,6,1} + 5\,\mathcal{C}_{3,6,2} - 4\,\mathcal{C}_{3,6,3} - 9\,\mathcal{C}_{3,6,4} - 3\,\mathcal{C}_{3,7,4} - 2\,\mathcal{C}_{3,7,5}\\
\mathcal{C}_{3,11,1} &= \frac{\lambda\,(-1+v)\,v\,\left(-648+6\,\lambda\,\left(544-41\,v+v^2\right)+\lambda^2\,\left(-3448+570\,v-39\,v^2+v^3\right)\right)\,\mathcal{C}_{0,0,1}}{216} +\\
&\quad \left(-58+\frac{\lambda\,\left(1544-119\,v+3\,v^2\right)}{12}\right)\,\mathcal{C}_{2,5,1} - 36\,\mathcal{C}_{3,6,1} - 2\,\mathcal{C}_{3,6,2} + 17\,\mathcal{C}_{3,6,3} + 24\,\mathcal{C}_{3,6,4} + 9\,\mathcal{C}_{3,7,4} + 5\,\mathcal{C}_{3,7,5}\\
\mathcal{C}_{3,12,1} &= \frac{\lambda\,(-1+v)\,v\,\left(6912-108\,\lambda\,\left(368-33\,v+v^2\right)+\lambda^2\,\left(48224-10312\,v+1017\,v^2-50\,v^3+v^4\right)\right)\,\mathcal{C}_{0,0,1}}{10368} +\\
&\quad \frac{\left(144-\lambda\,\left(368-33\,v+v^2\right)\right)\,\mathcal{C}_{2,5,1}}{12} - 2\,\mathcal{C}_{3,5,1} + 8\,\mathcal{C}_{3,6,1} - 4\,\mathcal{C}_{3,6,3} - 5\,\mathcal{C}_{3,6,4} - 2\,\mathcal{C}_{3,7,4} - \mathcal{C}_{3,7,5}
\end{aligned}
$$

History

RAB, EJF, 1994 Generating sets for m-line configurations of $1-(v, 2, \lambda)$ designs (i.e. regular graphs) for all m, v and λ.

Grannell, Griggs, Mendelsohn, 1995 Linear bases for m-line configurations of $2-(v, 3, 1)$ designs (i.e. Steiner triple systems) for $m \leq 4$ and all v.

Horak, Phillips, Wallis, Yucas Generating sets for m-line configurations in Steiner triple systems for all m and v. Linear bases for m-line configurations of Steiner triple systems with $m \leq 6$ and all v. Indicate that generating sets can be found for all k (i.e. Steiner systems).

Danziger, Grannell, Griggs, Mendelsohn, 1996 Explicit linear equations relating the fifty-six 5-block configurations of a Steiner triple system, valid for all v.

Urland Linear basis for 7-line configurations of Steiner triple systems, for all v.

RAB, 1997 Generating sets for m-line configurations of t-(v, k, λ) designs for all m, t, v, k and λ. Automated procedure to determine the linear equations explicitly.

References

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