

# An Introduction to Algebraic Graph Theory

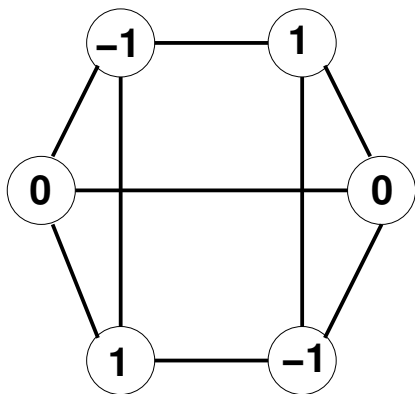
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Mathematics Department Seminar  
Reed College  
October 1, 2009

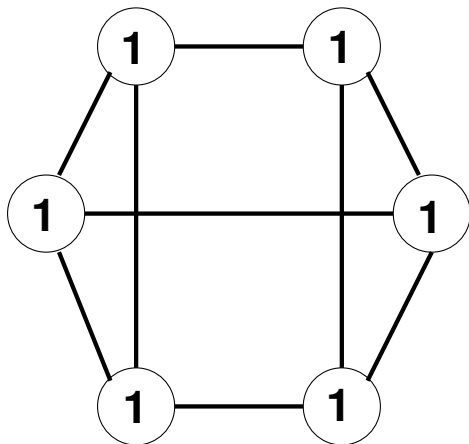
## Labeling Puzzles

- Assign a single real number value to each circle.
- For each circle, sum the values of adjacent circles.
- Goal:  
Sum at each circle should be a common multiple of the value at the circle.



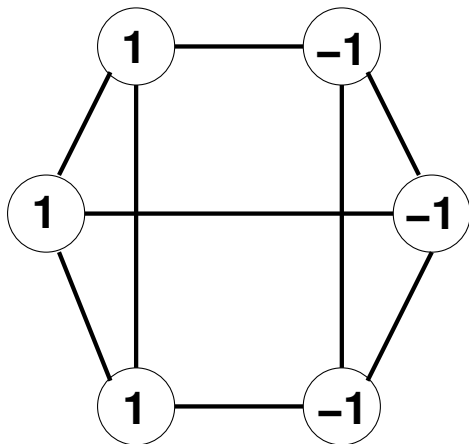
Common multiple:  $-2$

## Example Solutions



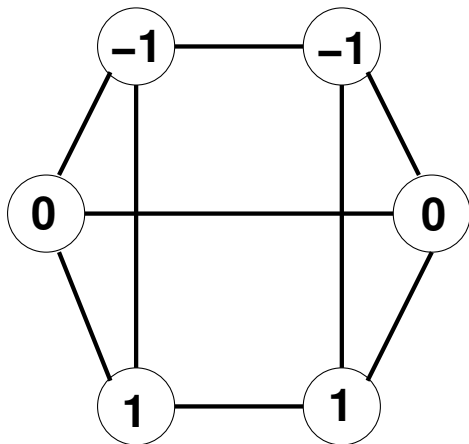
Common multiple: 3

## Example Solutions



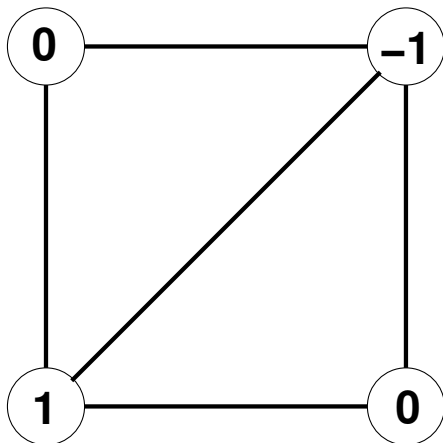
Common multiple: 1

## Example Solutions



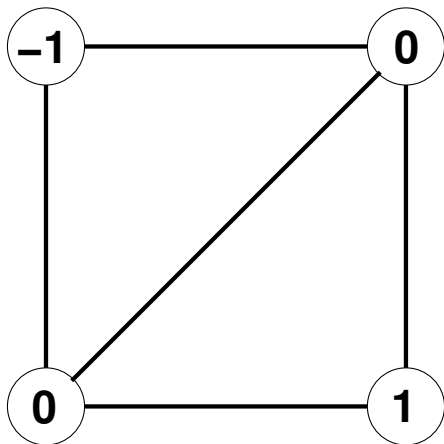
Common multiple: 0

## Example Solutions



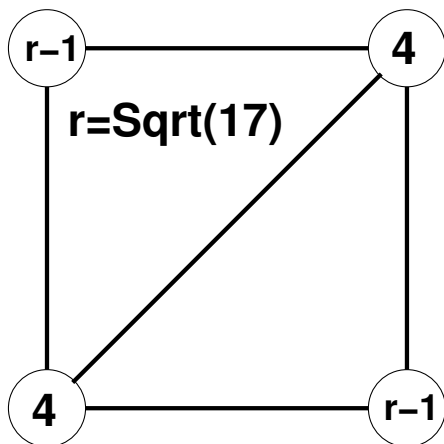
Common multiple:  $-1$

## Example Solutions



Common multiple: 0

## Example Solutions



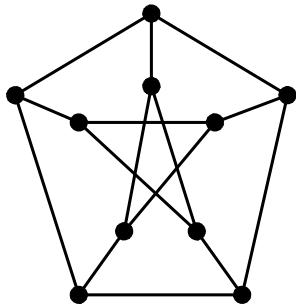
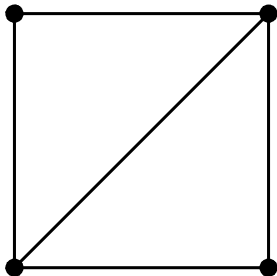
Common multiple:  $\frac{1}{2}(r+1) = \frac{1}{2}(\sqrt{17}+1)$



# Graphs

A **graph** is a collection of vertices (nodes, dots) where some pairs are joined by edges (arcs, lines).

The geometry of the vertex placement, or the contours of the edges are irrelevant. The relationships between vertices *are* important.



## Adjacency Matrix

Given a graph, build a matrix of zeros and ones as follows:

Label rows and columns with vertices, in the same order.

Put a 1 in an entry if the corresponding vertices are connected by an edge.

Otherwise put a 0 in the entry.

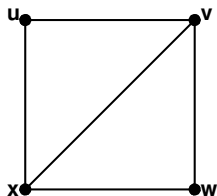
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	u	v	w	x
u	0	1	0	1
v	1	0	1	1
w	0	1	0	1
x	1	1	1	0

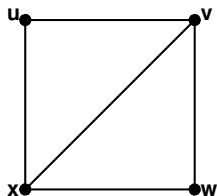
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Given a graph, build a matrix of zeros and ones as follows:

Label rows and columns with vertices, in the same order.

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Otherwise put a 0 in the entry.



- Always a symmetric matrix with zero diagonal.
- Useful for computer representations.
- Entrée to linear algebra, especially eigenvalues and eigenvectors.
- Symmetry groups of graphs is the other branch of Algebraic Graph Theory.

	u	v	w	x
u	0	1	0	1
v	1	0	1	1
w	0	1	0	1
x	1	1	1	0

# Eigenvalues of Graphs

$\lambda$  is an eigenvalue of a graph

$\Leftrightarrow \lambda$  is an eigenvalue of the adjacency matrix

$\Leftrightarrow A\vec{x} = \lambda\vec{x}$  for some vector  $\vec{x}$

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Adjacency matrix is real, symmetric  $\Rightarrow$

real eigenvalues, algebraic and geometric multiplicities are equal

minimal polynomial is product of linear factors for distinct eigenvalues

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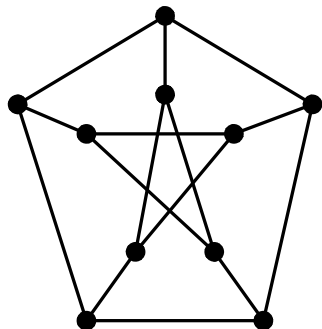
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Eigenvalues:

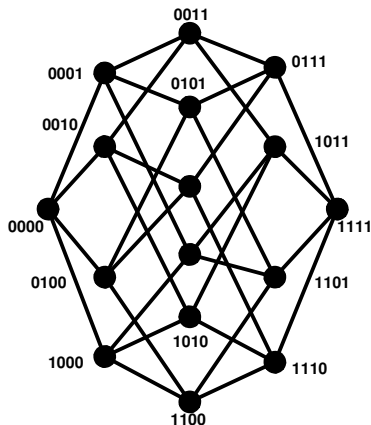
$$\lambda = 3 \qquad m = 1$$

$$\lambda = 1 \qquad m = 5$$

$$\lambda = -2 \qquad m = 4$$

# 4-Dimensional Cube

- Vertices: Length 4 binary strings
- Join strings differing in exactly one bit
- Generalizes 3-D cube



Eigenvalues:

$$\lambda = 4$$

$$m = 1$$

$$\lambda = 2$$

$$m = 4$$

$$\lambda = 0$$

$$m = 6$$

$$\lambda = -2$$

$$m = 4$$

$$\lambda = -4$$

$$m = 1$$



## Regular Graphs

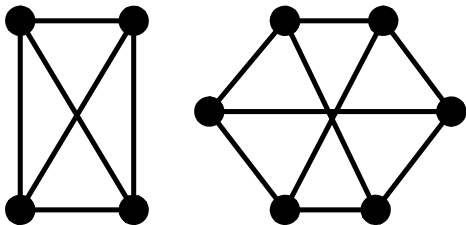
A graph is **regular** if every vertex has the same number of edges incident. The **degree** is the common number of incident edges.

### Theorem

Suppose  $G$  is a regular graph of degree  $r$ . Then

- $r$  is an eigenvalue of  $G$
- The multiplicity of  $r$  is the number of connected components of  $G$

Regular of degree 3  
with 2 components  
implies that  $\lambda = 3$   
will be an eigenvalue of  
multiplicity 2.



## Proof.

- Let  $\vec{u}$  be the vector where every entry is 1. Then

$$A\vec{u} = A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} r \\ r \\ \vdots \\ r \end{bmatrix} = r\vec{u}$$

- For each component of the graph, form a vector with 1's in entries corresponding to the vertices of the component, and zeros elsewhere.

These eigenvectors form a basis for the eigenspace of  $r$ .  
(Their sum is the vector  $\vec{u}$  above.)



## Labeling Puzzles Explained

The product of a graph's adjacency matrix with a column vector,  $A\vec{u}$ , forms sums of entries of  $\vec{u}$  for all adjacent vertices.

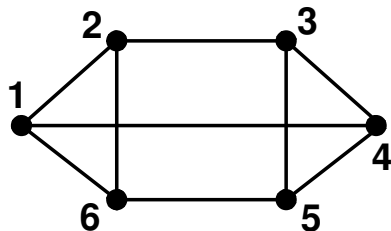
If  $\vec{u}$  is an eigenvector, then these sums should equal a common multiple of the numbers assigned to each vertex. This multiple is the **eigenvalue**.

So the puzzles earlier were simply asking for:

- eigenvectors (assignments of numbers to vertices), and
- eigenvalues (common multiples)

of the adjacency matrix of the graph.

# Eigenvalues and Eigenvectors of the Prism



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\lambda = 3$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

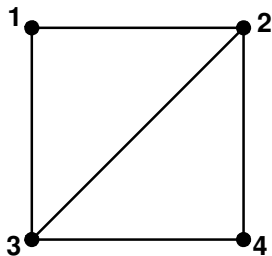
$\lambda = 0$

$$\begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = -2$

$$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

# Eigenvalues and Eigenvectors



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\lambda = -1$$

$$\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 0$$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = \frac{1}{2} (\sqrt{17} + 1)$$

$$\begin{bmatrix} \sqrt{17} - 1 \\ 4 \\ \sqrt{17} - 1 \\ 4 \end{bmatrix}$$

$$\lambda = \frac{1}{2} (-\sqrt{17} + 1)$$

$$\begin{bmatrix} \sqrt{17} + 1 \\ -4 \\ \sqrt{17} + 1 \\ -4 \end{bmatrix}$$

# Powers of Adjacency Matrices

## Theorem

*Suppose  $A$  is the adjacency matrix of a graph. Then the number of walks of length  $\ell$  between vertices  $v_i$  and  $v_j$  is the entry in row  $i$ , column  $j$  of  $A^\ell$ .*

# Powers of Adjacency Matrices

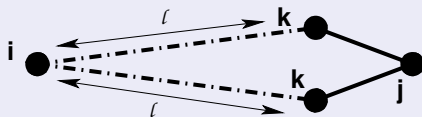
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## Proof.

Base case:  $\ell = 1$ . Adjacency matrix describes walks of length 1.

$$\begin{aligned} [A^{\ell+1}]_{ij} &= [A^\ell A]_{ij} = \sum_{k=1}^n [A^\ell]_{ik} [A]_{kj} = \sum_{k: v_k \text{ adj } v_j} [A^\ell]_{ik} \\ &= \text{total ways to walk in } \ell \text{ steps from } v_i \text{ to } v_k, \text{ a neighbor of } v_j \end{aligned}$$

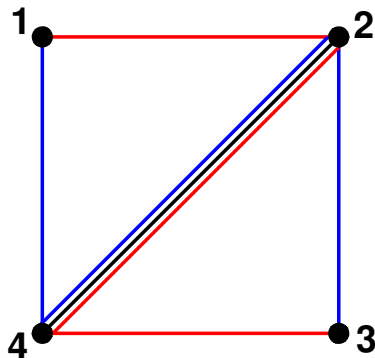


# Adjacency Matrix Power

Number of walks of length 3 between vertices 1 and 3?

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 5 & \boxed{2} & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{bmatrix}$$



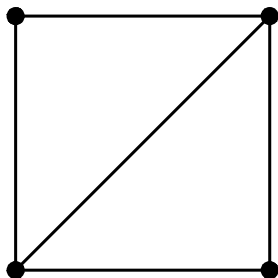


# Characteristic Polynomial

Recall the characteristic polynomial of a square matrix  $A$  is  $\det(\lambda I - A)$ .  
Roots of this polynomial are the eigenvalues of the matrix.

For the adjacency matrix of a graph on  $n$  vertices:

- Coefficient of  $\lambda^n$  is 1
- Coefficient of  $\lambda^{n-1}$  is zero (trace of adjacency matrix)
- Coefficient of  $-\lambda^{n-2}$  is number of edges
- Coefficient of  $-2\lambda^{n-3}$  is number of triangles



Characteristic polynomial  
 $\lambda^4 - 5\lambda^2 - 4\lambda + \dots$

## Diameter and Eigenvalues

The **diameter** of a graph is the longest shortest path. In other words, find the shortest route between each pair of vertices, then ask which pair is farthest apart?

### Theorem

*Suppose  $G$  is a graph of diameter  $d$ .  
Then  $G$  has at least  $d + 1$  distinct eigenvalues.*

### Proof.

Minimal polynomial is product of linear factors, one per distinct eigenvalue. There are zero short walks between vertices that are far apart. So the first  $d$  powers of  $A$  are linearly independent. So  $A$  cannot satisfy a polynomial of degree  $d$  or less. Thus minimal polynomial has at least  $d + 1$  factors hence at least  $d + 1$  eigenvalues. □

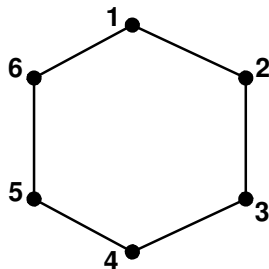
# Minimal Polynomial for a Diameter 3 Graph

Distinct Eigenvalues:  $\lambda = 2, 1, -1, -2$

Minimal Polynomial:

$$(x-2)(x-1)(x-(-1))(x-(-2)) = x^4 - 5x^2 + 4$$

Check that  $A^4 - 5A^2 + 4 = 0$



$A$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$A^2$

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

$A^3$

$$\begin{bmatrix} 0 & 3 & 0 & 2 & 0 & 3 \\ 3 & 0 & 3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 3 & 0 & 2 \\ 2 & 0 & 3 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 & 0 & 3 \\ 3 & 0 & 2 & 0 & 3 & 0 \end{bmatrix}$$

## Results Relating Diameters and Eigenvalues

Regular graph:  $n$  vertices, degree  $r$ , diameter  $d$

Let  $\lambda$  denote second largest eigenvalue ( $r$  is largest eigenvalue)

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- Alon, Milman, 1985

$$d \leq 2 \left\lceil \left( \frac{2r}{r - \lambda} \right)^{\frac{1}{2}} \log_2 n \right\rceil$$

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- Mohar, 1991

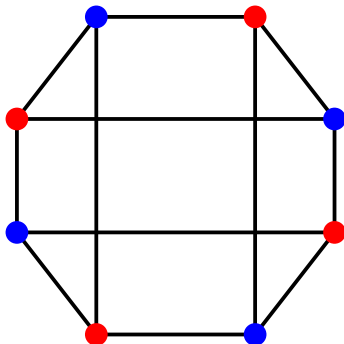
$$d \leq 2 \left\lceil \left( \frac{2r - \lambda}{4(r - \lambda)} \right) \ln(n - 1) \right\rceil$$

# Bipartite Graphs

A graph is **bipartite** if the vertex set can be split into two parts, so that every edge goes from part to the other. Identical to being “two-colorable.”

## Theorem

*A graph is bipartite if and only if there are no cycles of odd length.*



# Eigenvalues of Bipartite Graphs

## Theorem

Suppose  $G$  is a bipartite graph with eigenvalue  $\lambda$ .  
Then  $-\lambda$  is also an eigenvalue of  $G$ .

$$\lambda = 3$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda = -3$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\lambda = 1$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = -1$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$



## Proof.

Order the vertices according to the two parts. Then

$$A = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix}$$

For an eigenvector  $\vec{u} = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix}$  of  $A$  for  $\lambda$ ,

$$\begin{bmatrix} \lambda \vec{u}_1 \\ \lambda \vec{u}_2 \end{bmatrix} = \lambda \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = \lambda \vec{u} = A\vec{u} = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = \begin{bmatrix} B\vec{u}_2 \\ B^t\vec{u}_1 \end{bmatrix}$$

Now set  $\vec{v} = \begin{bmatrix} -\vec{u}_1 \\ \vec{u}_2 \end{bmatrix}$  and compute,

$$A\vec{v} = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix} \begin{bmatrix} -\vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = \begin{bmatrix} B\vec{u}_2 \\ -B^t\vec{u}_1 \end{bmatrix} = \begin{bmatrix} \lambda \vec{u}_1 \\ -\lambda \vec{u}_2 \end{bmatrix} = -\lambda \begin{bmatrix} -\vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = -\lambda \vec{v}$$

# Bipartite Graphs on an Odd Number of Vertices

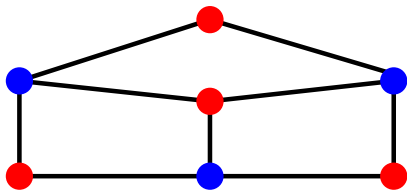
## Theorem

Suppose  $G$  is a bipartite graph with an odd number of vertices.  
Then zero is an eigenvalue of the adjacency matrix of  $G$ .

## Proof.

Pair each eigenvalue with its negative.

At least one eigenvalue must then equal its negative. □



$$\lambda = \sqrt{7} \quad m = 1$$

$$\lambda = 1 \quad m = 2$$

$$\lambda = 0 \quad m = 1$$

$$\lambda = -1 \quad m = 2$$

$$\lambda = -\sqrt{7} \quad m = 1$$

## Application: Friendship Theorem

### Theorem

*Suppose that at a party every pair of guests has exactly one friend in common. Then there are an odd number of guests, and one guest knows everybody.*

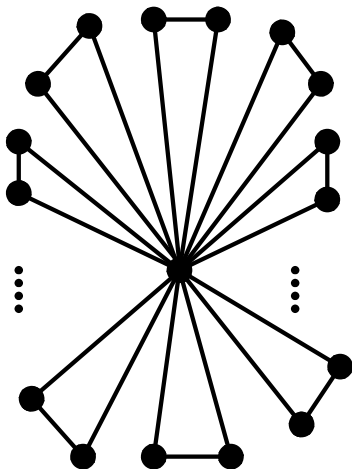
- Model party with a graph
- Vertices are guests, edges are friends
- Two possibilities:

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- Model party with a graph
- Vertices are guests, edges are friends
- Two possibilities:
  1. Graph to the right
  2. Algebraic graph theory says the adjacency matrix has eigenvalues with non-integer multiplicities



## Application: Buckminsterfullerene

- Molecule composed solely of 60 carbon atoms
- “Carbon-60,” “buckyball”
- At each atom two single bonds and one double bond describe a graph that is regular of degree 3
- Graph is the skeleton of a truncated icosahedron, a soccer ball
- $60 \times 60$  adjacency matrix
- Eigenvalues of adjacency matrix predict peaks in mass spectrometry
- First created in 1985 (1996 Nobel Prize in Chemistry)

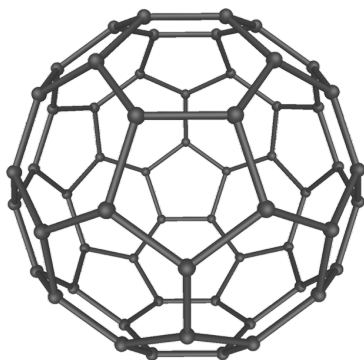
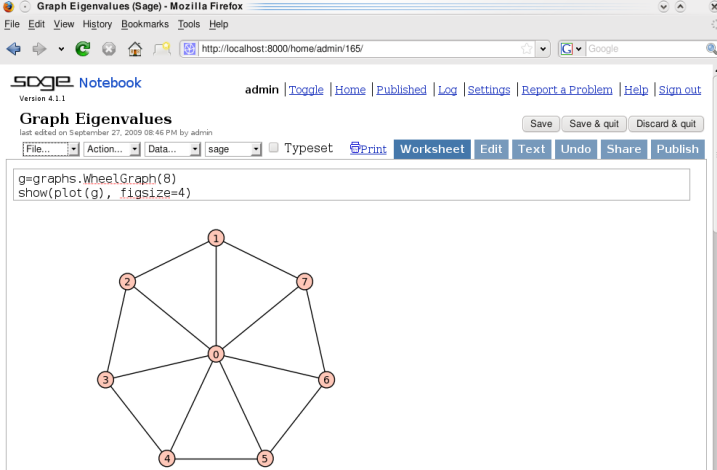


Illustration by Michael Ströck  
Wikipedia, GFDL License

# Sage and Graphs

Sage is software for mathematics, “creating a viable free open source alternative to Magma, Maple, Mathematica and Matlab.”



The screenshot shows a web browser window titled "Graph Eigenvalues (Sage) - Mozilla Firefox". The address bar shows the URL `http://localhost:8000/home/admin/165/`. The page header includes "SAGE Notebook Version 4.1.1" and navigation links for "admin", "Toggle", "Home", "Published", "Log", "Settings", "Report a Problem", "Help", and "Sign out". The main content area is titled "Graph Eigenvalues" and shows the code `g=graphs.WheelGraph(8)` and `show(plot(g, figsize=4))`. Below the code is a plot of a wheel graph with 8 vertices and 15 edges. The vertices are labeled 0 through 7, with vertex 0 at the center and vertices 1 through 7 forming an outer cycle. The plot is displayed in a window with a title bar and standard OS controls.

# Sage and Graph Eigenvalues

```
print g.spectrum()
print g.eigenspaces()

[3.828427124746190?, 1.246979603717467?, 1.246979603717467?,
-0.4450418679126288?, -0.4450418679126288?, -1.801937735804839?,
-1.801937735804839?, -1.828427124746191?]
[
(a0, Vector space of degree 8 and dimension 1 over Number Field in a0
with defining polynomial x^2 - 2*x - 7
User basis matrix:
[ 1 1/7*a0 1/7*a0 1/7*a0 1/7*a0 1/7*a0 1/7*a0 1/7*a0]),
(a1, Vector space of degree 8 and dimension 2 over Number Field in a1
with defining polynomial x^3 + x^2 - 2*x - 1
User basis matrix:
[ 0 1 0 -1 -a1 -a1^2 + 1 a1^2 - 1
a1]
[ 0 0 1 a1 a1^2 - 1 -a1^2 + 1 -a1
-1])
]
```

```
f(x)=x^2 - 2*x - 7
g(x)=x^3 + x^2 - 2*x - 1
print f(-1.828427124746191)
print g(1.246979603717467)
```

```
4.88498130835069e-15
-4.44089209850063e-16
```

# Isospectral, Non-Isomorphic Circulant Graphs

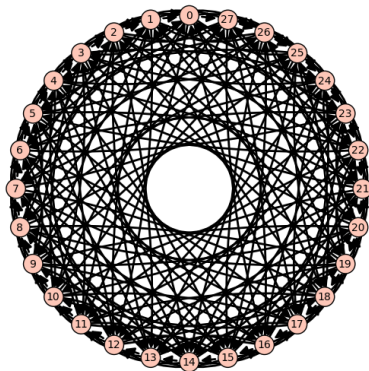
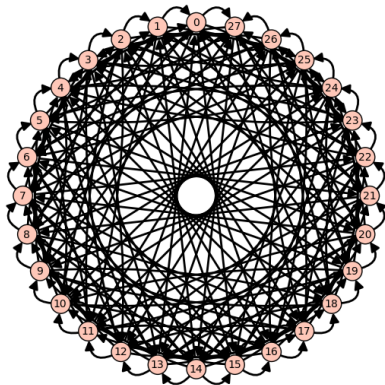
- A graph is **circulant** if each row of the adjacency matrix is a shift (by one) of the prior row.



# Isospectral, Non-Isomorphic Circulant Graphs

- A graph is **circulant** if each row of the adjacency matrix is a shift (by one) of the prior row.
- Julia Brown (nee Lazenby), 2008 Reed thesis
- Infinite family of pairs of digraphs:  
identical spectra, but non-isomorphic graphs
- Vertices number  $n = 2^r p$ , with  $r \geq 2$ ,  $p$  prime

$$n = 2^2 \cdot 7$$



Sage: 6 lines to construct, 2 lines to plot, 2 lines to verify  
Also verified for  $n = 2^5 \cdot 13 = 416$  (in under 6 seconds)

## Moral of the Story

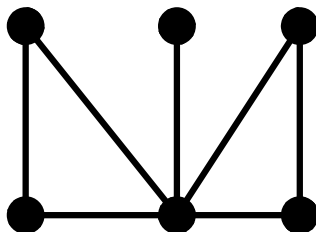
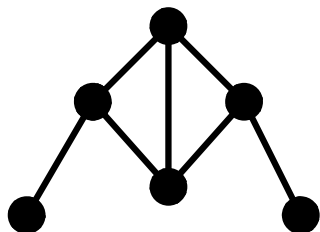
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(regular of degree  $r \Rightarrow r$  is an eigenvalue)

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- Linear algebra properties predict graph properties  
(functions of two largest eigenvalues bound the diameter)

## Moral of the Story

- Graph properties predict linear algebra properties  
(regular of degree  $r \Rightarrow r$  is an eigenvalue)
- Linear algebra properties predict graph properties  
(functions of two largest eigenvalues bound the diameter)
- Do eigenvalues characterize graphs? No.



# Bibliography

- Norman Biggs, *Algebraic Graph Theory*, Second Edition  
Cambridge Mathematical Library, 1974, 1993
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