Solving Sudoku with Dancing Links

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October 25, 2010

Available at http://buzzard.pugetsound.edu/talks.html

Example: Combinatorial Enumeration

Create all permutations of the set $\{0, 1, 2, 3\}$

- Simple example to demonstrate key ideas
- Creation, cardinality, existence?
- There are more efficient methods for this example

Brute Force Backtracking

BLACK = Forward BLUE = Solution RED = Backtrack

root	0 1 2	0 1 3 3	0 2 1	0 2 3 3	0 3 1	0 3	1 0 2
0	0 1 2 2	0 1 3	0 2 1 3	0 2 3	0 3 1 3	0 3 3	1 0 2 1
0 0	0 1 2	0 1	0 2 1	0 2	0 3 1	0 3	1 0 2
0	0 1 2 3	0	0 2	0	0 3	0	1 0 2 2
0 1	0 1 2	0 2	0 2 2	0 3	0 3 2	root	1 0 2
0 1 0	0 1	0 2 0	0 2	0 3 0	0 3 2 0	1	1 0 2 3
0 1	0 1 3	0 2	0 2 3	0 3	0 3 2	1 0	1 0 2
0 1 1	0 1 3 0	0 2 1	0 2 3 0	0 3 1	0 3 2 1	1 0 0	⋮
0 1	0 1 3	0 2 1 0	0 2 3	0 3 1 0	0 3 2	1 0	⋮
0 1 2	0 1 3 1	0 2 1	0 2 3 1	0 3 1	0 3 2 2	1 0 1	⋮
0 1 2 0	0 1 3	0 2 1 1	0 2 3	0 3 1 1	0 3 2	1 0	⋮
0 1 2	0 1 3 2	0 2 1	0 2 3 2	0 3 1	0 3 2 3	1 0 2	⋮
0 1 2 1	0 1 3	0 2 1 2	0 2 3	0 3 1 2	0 3 2	1 0 2 0	⋮

A Better Idea

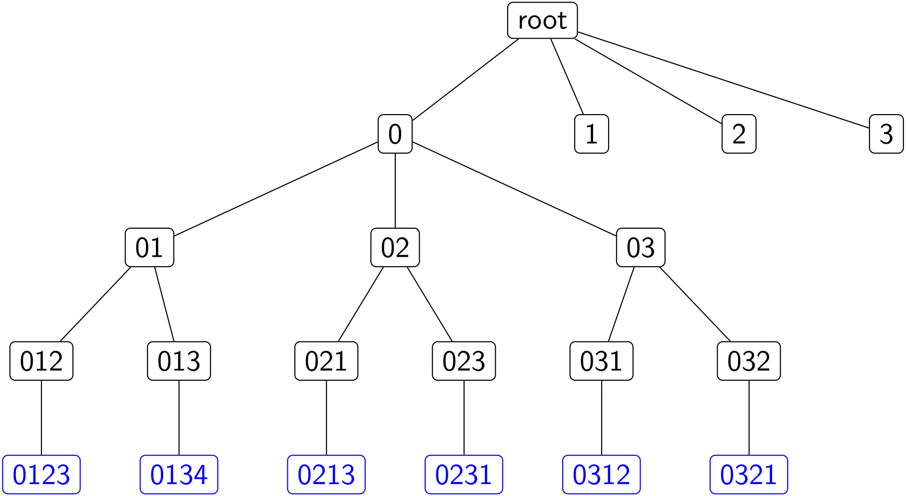
- Avoid the really silly situations, such as: 1 0 1
- "Remember" that a symbol has been used already
- Additional data structure: track "available" symbols
- Critical: must maintain this extra data properly
- (Note recursive nature of backtracking)

Sophisticated Backtracking

BLACK = Forward　　　BLUE = Solution　　　RED = Backtrack

root {0,1,2,3}	0 2 1 3 {}	0 3 2 1 {}	1 0 {2,3}	1 3 0 2 {}
0 {1,2,3}	0 2 1 {3}	0 3 2 {1}	1 {0,2,3}	1 3 0 {2}
0 1 {2,3}	0 2 {1,3}	0 3 {1,2}	1 2 {0,3}	1 3 {0,2}
0 1 2 {3}	0 2 3 {1}	0 {1,2,3}	1 2 0 {3}	1 3 2 {0}
0 1 2 3 {}	0 2 3 1 {}	root {0,1,2,3}	1 2 0 3 {}	1 3 2 0 {}
0 1 2 {3}	0 2 3 {1}	1 {0,2,3}	1 2 0 {3}	1 3 2 {0}
0 1 {2,3}	0 2 {1,3}	1 0 {2,3}	1 2 {0,3}	1 3 {0,2}
0 1 3 {2}	0 {1,2,3}	1 0 2 {3}	1 2 3 {0}	1 {0,2,3}
0 1 3 2 {}	0 3 {1,2}	1 0 2 3 {}	1 2 3 0 {}	root {0,1,2,3}
0 1 3 {2}	0 3 1 {2}	1 0 2 {3}	1 2 3 {0}	2 {0,1,3}
0 1 {2,3}	0 3 1 2 {}	1 0 {2,3}	1 2 {0,3}	⋮
0 {1,2,3}	0 3 1 {2}	1 0 3 {2}	1 {0,2,3}	⋮
0 2 {1,3}	0 3 {1,2}	1 0 3 2 {}	1 3 {0,2}	⋮
0 2 1 {3}	0 3 2 {1}	1 0 3 {2}	1 3 0 {2}	

Depth-First Search Tree



Algorithm

```
n=4
available=[True]*n   # [True, True, True, True]
perm=[0]*n           # [0, 0, 0, 0]

def bt(level):
    for x in range(n):
        if available[x]:
            available[x]=False
            perm[level]=x
            if level+1 == n:
                print perm
            bt(level+1)
            available[x]=True

bt(0)
```

Sudoku Basics

- n^2 symbols
- $n^2 \times n^2$ grid
- n^2 subgrids ("boxes") each $n \times n$
- Classic Sudoku is $n = 3$
- Each symbol once and only once in each row
- Each symbol once and only once in each column
- Each symbol once and only once in each box
- The grid begins partially completed
- A Sudoku puzzle should have a unique completion

Example

5				8			4	9
			5				3	
	6	7	3					1
1	5							
			2		8			
							1	8
7					4	1	5	
	3				2			
4	9			5				3

$\Longrightarrow$

5	1	3	6	8	7	2	4	9
8	4	9	5	2	1	6	3	7
2	6	7	3	4	9	5	8	1
1	5	8	4	6	3	9	7	2
9	7	4	2	1	8	3	6	5
3	2	6	7	9	5	4	1	8
7	8	2	9	3	4	1	5	6
6	3	5	1	7	2	8	9	4
4	9	1	8	5	6	7	2	3

Sudoku via Backtracking

- Fill in first row, left to right, then second row, . . .
- For each blank cell, maintain possible new entries
- As entries are attempted, update possibilities
- If a cell has just one possibility, it is forced
- Lots to keep track of, especially at backtrack step

Sudoku via Backtracking

- Fill in first row, left to right, then second row, . . .
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- Alternate Title: “Why I Don’t Do Sudoku”

Top row, second column: possibilities?

5				8			4	9
			5				3	
	6	7	3					1
1	5							
			2		8			
							1	8
7					4	1	5	
	3				2			
4	9			5				3

$\{1,2,3,6,7\}$

$$\{1,2,4,7,8\} \longrightarrow \{1,2,4,7,8\} \cap \{1,2,3,6,7\} = \{1,2,7\}$$

Suppose we try 2 first.
Seventh row, second column: possibilities?

5	2			8			4	9
			5				3	
	6	7	3					1
1	5							
			2		8			
							1	8
7					4	1	5	
	3				2			
4	9			5				3

$\{2,3,6,8,9\}$

$$\{1,4,7,8\} \longrightarrow \{1,4,7,8\} \cap \{2,3,6,8,9\} = \{8\}$$

One choice!
This may lead to other singletons in the affected row or column.

Exact Cover Problem

- Given: matrix of 0's and 1's
- Find: subset of rows
- Condition: rows sum to exactly the all-1's vector
- Amenable to backtracking (on columns, not rows!)
- Example: (Knuth)

$$\begin{matrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{matrix}$$

Solution

Select rows 1, 4 and 5:

$$\begin{array}{cccccccc}
\color{blue}{\Rightarrow} & 0 & 0 & \color{blue}{1} & 0 & \color{blue}{1} & \color{blue}{1} & 0 \\
 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
\color{blue}{\Rightarrow} & \color{blue}{1} & 0 & 0 & \color{blue}{1} & 0 & 0 & 0 \\
\color{blue}{\Rightarrow} & 0 & \color{blue}{1} & 0 & 0 & 0 & 0 & \color{blue}{1} \\
 & 0 & 0 & 0 & 1 & 1 & 0 & 1
\end{array}$$

$$\Big\downarrow$$

$$\color{blue}{1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1}$$

Sudoku as an Exact Cover Problem

- Matrix rows are per symbol, per grid location $(n^2 \times (n^2 \times n^2) = n^6)$
- Matrix columns are conditions: $(3n^4$ total)
  - Per symbol, per grid row: symbol in row $(n^2 \times n^2)$
  - Per symbol, per grid column: symbol in column $(n^2 \times n^2)$
  - Per symbol, per grid box: symbol in box $(n^2 \times n^2)$

Place a 1 in entry of the matrix
if and only if
matrix row describes symbol placement satisfying matrix column condition

- Example:
  Consider matrix row that places a 7 in grid at row 4, column 9
  - 1 in matrix column for "7 in grid row 4"
  - 1 in matrix column for "7 in grid column 9"
  - 1 in matrix column for "7 in grid box 6"
  - 0 elsewhere

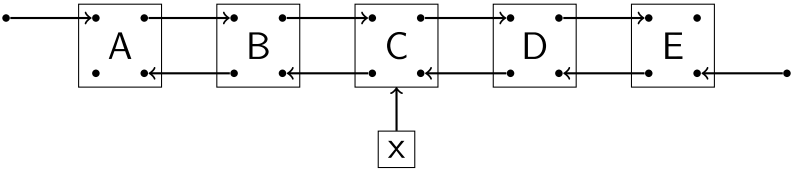
Sudoku as an Exact Cover Problem

- Puzzle is “pre-selected” matrix rows
- Can delete these matrix rows, and their “covered matrix columns”
- $n = 3$: 729 matrix rows, 243 matrix columns
- Previous example: Remove 26 rows, remove $3 \times 26 = 78$ columns
- Select $81 - 26 = 55$ rows, from 703, for exact cover (uniquely)
- Selected rows describe placement of symbols into locations for Sudoku solution

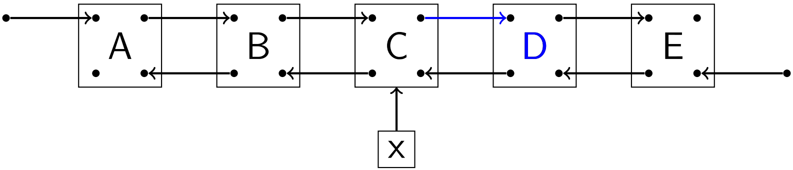
Dancing Links

- Manage lists with frequent deletions and restorations
- Perfect for descending, backtracking in a search tree
- Hitotumatu, Noshita (1978, Information Processing Letters)
  - "pointers of each already-used element are still active while... removed"
  - Two pages, *N* queens problem
  - Donald Knuth listed in the Acknowledgement
- Popularized by Knuth, "Dancing Links" (2000, arXiv)
  - Algorithm X = "traditional" backtracking
  - Algorithm DLX = Dancing Links + Algorithm X
  - 26 pages, applications to packing pentominoes in a square

Doubly-Linked List

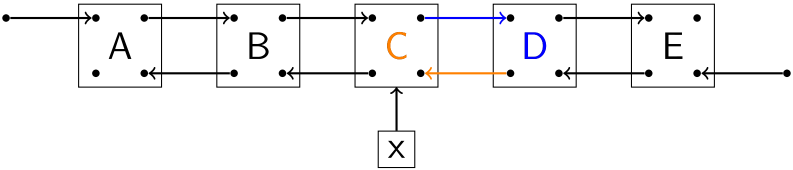


Remove Node "C" From List



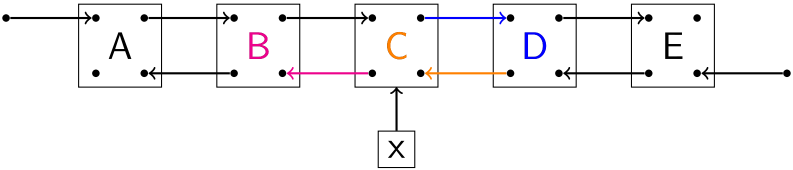
$$R[x]$$

Remove Node "C" From List



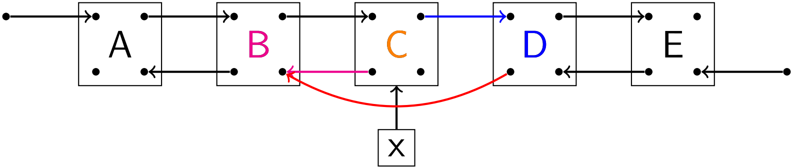
$$L[R[x]]$$

Remove Node "C" From List



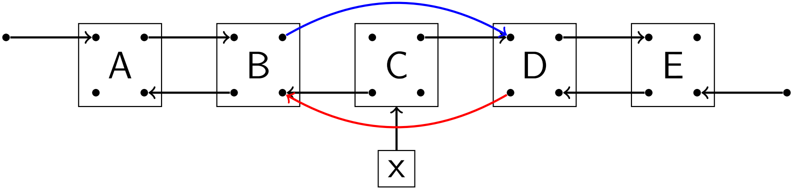
L[R[x]] L[x]

Remove Node “C” From List



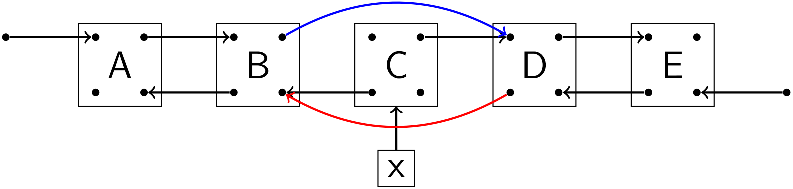
$$L[R[x]] \longleftarrow L[x]$$

Two Assignments to Totally Remove "C"



$L[R[x]] \longleftarrow L[x]$ $R[L[x]] \longleftarrow R[x]$

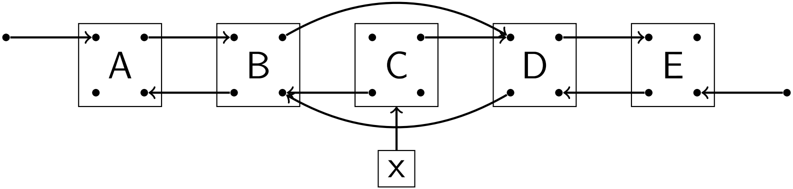
Two Assignments to Totally Remove "C"



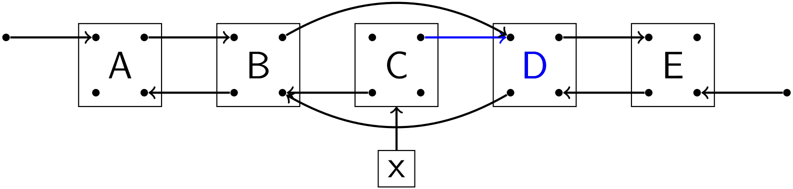
$$L[R[x]] \longleftarrow L[x] \qquad R[L[x]] \longleftarrow R[x]$$

DO NOT CLEAN UP THE MESS

List Without "C", Includes Our Mess

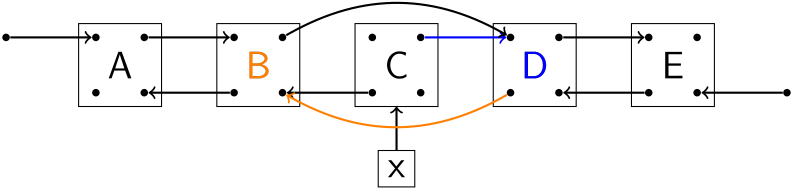


Restore Node "C" to the List



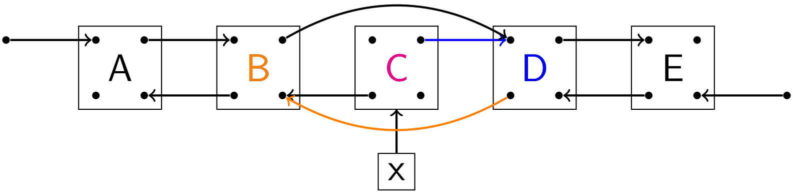
R[x]

Restore Node "C" to the List



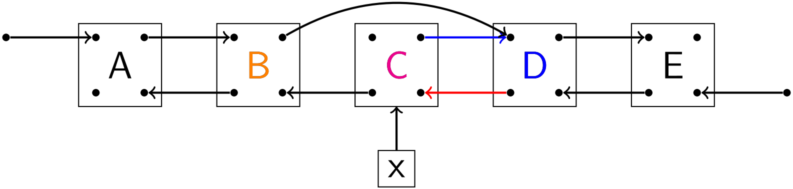
$L[R[x]]$

Restore Node "C" to the List



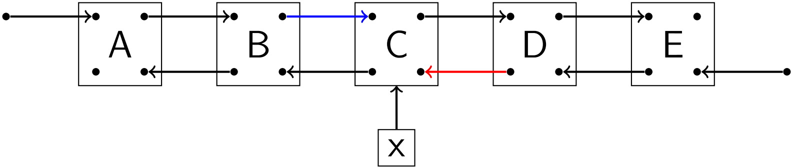
L[R[x]] x

Restore Node "C" to the List



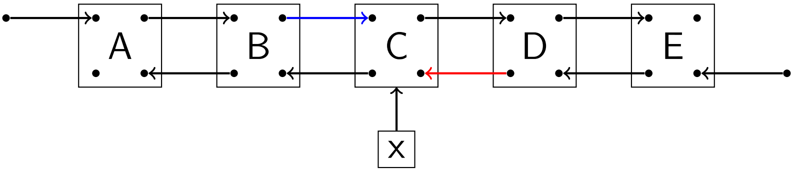
$$L[R[x]] \longleftarrow x$$

Restore Node "C" to the List



$L[R[x]] \longleftarrow x$ $\qquad$ $R[L[x]] \longleftarrow x$

Restore Node “C” to the List



$L[R[x]] \longleftarrow x$ $\qquad$ $R[L[x]] \longleftarrow x$

WE NEED OUR MESS, IT CLEANS UP ITSELF

DLX for the Exact Cover Problem

- Backtrack on the columns
- Choose a column to cover, this will dictate a selection of rows

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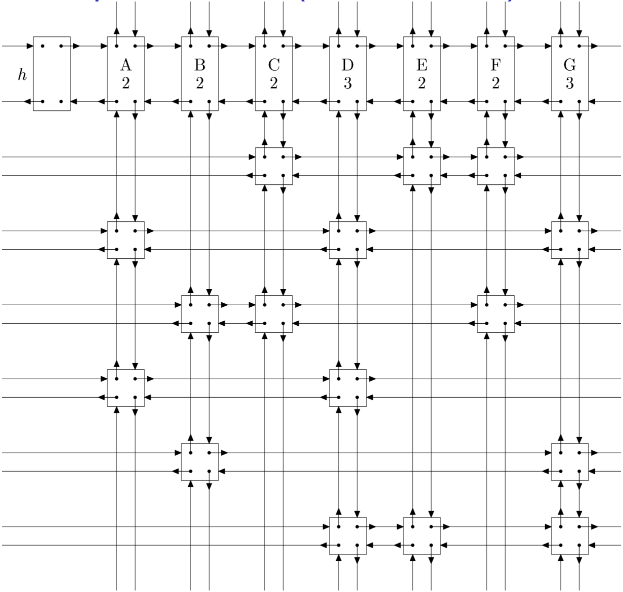
DLX for the Exact Cover Problem

- Backtrack on the columns
- Choose a column to cover, this will dictate a selection of rows
- Loop over rows, for each row choice remove covered columns
- Recursively analyze new, smaller matrix
- Restore rows and columns on backtrack step

Exact Cover Example (Knuth, 2000)

	A	B	C	D	E	F	G
1	0	0	1	0	1	1	0
2	1	0	0	1	0	0	1
3	0	1	1	0	0	1	0
4	1	0	0	1	0	0	0
5	0	1	0	0	0	0	1
6	0	0	0	1	1	0	1

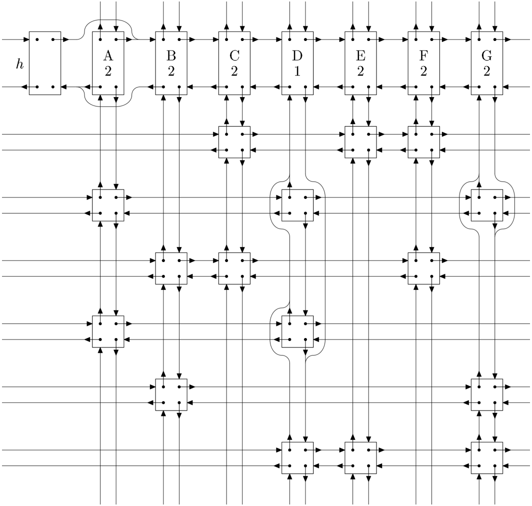
Exact Cover Representation (Knuth, 2000)



Exact Cover Representation (Knuth, 2000)

- Cover column A
- Remove rows 2, 4

	A	B	C	D	E	F	G
1	0	0	1	0	1	1	0
2	1	0	0	1	0	0	1
3	0	1	1	0	0	1	0
4	1	0	0	1	0	0	0
5	0	1	0	0	0	0	1
6	0	0	0	1	1	0	1

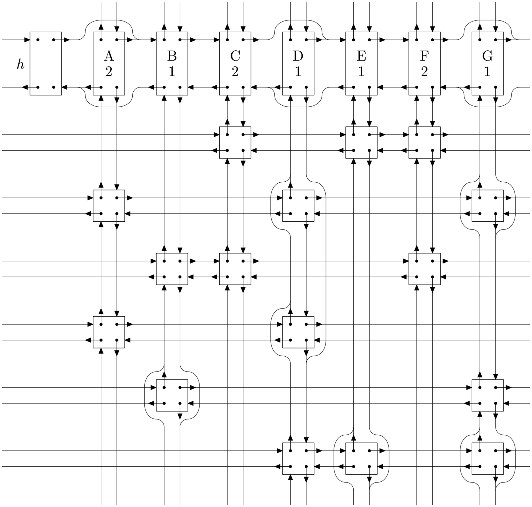


Exact Cover Representation (Knuth, 2000)

- Loop through rows
- Row 2 covers D, G
- D removes row 4, 6
- G removes row 5, 6

	A	B	C	D	E	F	G
1	0	0	1	0	1	1	0
2	1	0	0	1	0	0	1
3	0	1	1	0	0	1	0
4	1	0	0	1	0	0	0
5	0	1	0	0	0	0	1
6	0	0	0	1	1	0	1

Recurse on 2×4 matrix
It has no solution,
so will soon backtrack



Implementation in Sage

> The games module only contains code for solving Sudoku puzzles, which I wrote in two hours on Alaska Airlines, in order to solve the puzzle in the inflight magazine. — William Stein, Sage Founder

- Sage, open source mathematics software, sagemath.org

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- Sage, open source mathematics software, sagemath.org
- Stein (UW): naive recursive backtracking, run times of 30 minutes
- Carlo Hamalainen (Turkey/Oz): DLX for exact cover problems
- Tom Boothby (UW): Preliminary representation as an exact cover
- RAB: Optimized backtracking
  - lots of look-ahead
  - automatic Cython conversion of Python to C
- RAB: new class, conveniences for printing, finished DLX approach

Timings in Sage

Test Examples:

- Original doctest, provenance is Alaska Airlines in-flight magazine?
- 17-hint "random" puzzle (no 16-hint puzzle known)
- Worst-case: top-row empty, top-row solution 9 8 7 6 5 4 3 2 1
- All ∼48,000 known 17-hint puzzles (Gordon Royle, UWA)

Equipment: R 3500 machine, 3 GHz Intel Core Duo

Puzzle	Time (milliseconds)		
	Naive	Custom	DLX
Alaska	34	0.187	1.11
17	1,494,000	441.0	1.20
Worst	4,798,000	944.0	1.21
48K 17			∼60,000

Talk available at:

buzzard.pugetsound.edu/talks.html