THE ROLE OF TECHNOLOGY IN A FIRST LINEAR ALGEBRA COURSE

Robert A. Beezer

University of Puget Sound

SIAM CONFERENCE ON APPLIED LINEAR ALGEBRA

MAY 17, 2021

“TECHNOLOGY”

When we say “technology” for teaching I think we mean many different things. (And we *hear* different things!)

Parallel: “data”, as in “Social media sites abuse your DATA.”

- Personally identifiable information? Identification (SSN)? Financial information (account numbers)? Health records?
- Google: Maps, numerous discussion boards, Discover application (news feed)
- Facebook: your social graph, your friends

What do we mean by “technology” for teaching linear algebra?

FOUR TYPES OF TECHNOLOGY

- Tutorial Assistants
  - Reduced Row-Echelon Form
  - `www.math.odu.edu/~bogacki/lat/`
- Point-and-Click Interactive Explorations
  - David Austin's Understanding Linear Algebra
  - Matrices as linear transformations
  - Eigenvalues and eigenvectors is a similar class
- Computationally-Assisted Demonstrations
  - SVD Rank-One Decomposition for image compression
  - A Sage `interact` hosted on CoCalc.com
- Computationally-Assisted Explorations
  - Eigenvalues of a Companion Matrix
  - Simulated exam question via Sage Cell Server

EIGENVALUES WITHOUT DETERMINANTS

1. For $n \times n$ matrix A, choose *any* nonzero vector $\mathbf{x} \in \mathbb{C}^n$.
2. Then we have a linearly dependent set

$$A^0\mathbf{x},\ A^1\mathbf{x},\ A^2\mathbf{x}, \dots,\ A^n\mathbf{x}$$

3. And a relation of linear dependence

$$a_0 A^0\mathbf{x} + a_1 A^1\mathbf{x} + a_2 A^2\mathbf{x} + \cdots + a_n A^n\mathbf{x} = \mathbf{0}$$

4. Define a polynomial and factor (over $\mathbb{C}$!):

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = (x - \lambda_1)(x - \lambda_2)\dots(x - \lambda_n)$$

5. And thus

$$\mathbf{0} = p(A)\mathbf{x} = (A - \lambda_1 I)(A - \lambda_2 I)\dots(A - \lambda_n I)\mathbf{x}$$

6. Build up the product from right to left. At some point a nonzero vector is annihilated by $(A - \lambda_k I)$.
7. λ_k is an eigenvalue, the last nonzero product is the eigenvector.

EIGENVALUES OF A COMPANION MATRIX

Companion matrix, A, and its first five powers:

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A = A^1 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & -5 & -3 \\ 1 & 0 & 3 & -2 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 2 & 2 & 8 \\ 0 & -5 & -3 & -18 \\ 0 & 3 & -2 & 9 \\ 1 & 1 & 4 & 2 \end{bmatrix} \quad A^4 = \begin{bmatrix} 2 & 2 & 8 & 4 \\ -5 & -3 & -18 & -2 \\ 3 & -2 & 9 & -12 \\ 1 & 4 & 2 & 11 \end{bmatrix}$$

First column is predictable, so choose $\mathbf{x} = (1,\, 0,\, 0,\, 0)^t$.

First four initial columns are linearly independent, first five are clearly linearly dependent.

Relation of linear dependence: $-2A^0\mathbf{x} + 5A^1\mathbf{x} - 3A^2\mathbf{x} - 1A^3\mathbf{x} + A^4\mathbf{x} = \mathbf{0}$.

$p(x) = -2 + 5x - 3x^2 - x^3 + x^4 = (x+2)(x-1)^3$

CONCLUSION

- "Technology" in teaching (linear algebra) can mean lots of things.
- Linear algebra is a *perfect* disipline for using technology:
  - unimaginable dimensions (greater than 4)
  - impossible "by-hand" computations (10th power of a 10×10 matrix)
- We owe it to students with applied interests in the modern world to expose them to computation.
- Computational power, easily available in our modern world, can illustrate
  - applications, such as SVD image compression), *and*
  - theory, such as eigenvalues of companion matrices

Links

- Slides: `buzzard.ups.edu/talks.html`
- Sage-Enabled Textbook: A First Course in Linear Algebra, `linear.ups.edu`