THE ROLE OF TECHNOLOGY IN A FIRST LINEAR ALGEBRA COURSE

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When we say “technology” for teaching I think we mean many different things. (And we hear different things!)

Parallel: “data”, as in “Social media sites abuse your DATA.”

- Personally identifiable information? Identification (SSN)? Financial information (account numbers)? Health records?
- Google: Maps, numerous discussion boards, Discover application (news feed)
- Facebook: your social graph, your friends

What do we mean by “technology” for teaching linear algebra?
FOUR TYPES OF TECHNOLOGY

- Tutorial Assistants
  - Reduced Row-Echelon Form
  - [www.math.odu.edu/~bogacki/lat/](http://www.math.odu.edu/~bogacki/lat/)

- Point-and-Click Interactive Explorations
  - David Austin's *Understanding Linear Algebra*
  - Matrices as linear transformations
  - Eigenvalues and eigenvectors is a similar class

- Computationally-Assisted Demonstrations
  - SVD Rank-One Decomposition for image compression
  - A Sage interact hosted on [CoCalc.com](http://CoCalc.com)

- Computationally-Assisted Explorations
  - Eigenvalues of a Companion Matrix
  - Simulated exam question via [Sage Cell Server](http://Sage Cell Server)
EIGENVALUES WITHOUT DETERMINANTS

1. For \( n \times n \) matrix \( A \), choose any nonzero vector \( x \in \mathbb{C}^n \).

2. Then we have a linearly dependent set

\[
A^0 x, A^1 x, A^2 x, \ldots, A^n x
\]

3. And a relation of linear dependence

\[
a_0 A^0 x + a_1 A^1 x + a_2 A^2 x + \cdots + a_n A^n x = 0
\]

4. Define a polynomial and factor (over \( \mathbb{C}! \)):

\[
p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = (x - \lambda_1)(x - \lambda_2)\ldots(x - \lambda_n)
\]

5. And thus

\[
0 = p(A)x = (A - \lambda_1 I)(A - \lambda_2 I)\ldots(A - \lambda_n I)x
\]

6. Build up the product from right to left. At some point a nonzero vector is annihilated by \((A - \lambda_k I)\).

7. \( \lambda_k \) is an eigenvalue, the last nonzero product is the eigenvector.
EIGENVALUES OF A COMPANION MATRIX

Companion matrix, $A$, and its first five powers:

$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A = A^1 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & -5 & -3 \\ 1 & 0 & 3 & -2 \\ 0 & 1 & 1 & 4 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 2 & 2 & 8 \\ 0 & -5 & -3 & -18 \\ 0 & 3 & -2 & 9 \\ 1 & 1 & 4 & 2 \end{bmatrix}$

$A^4 = \begin{bmatrix} 2 & 2 & 8 & 4 \\ -5 & -3 & -18 & -2 \\ 3 & -2 & 9 & -12 \\ 1 & 4 & 2 & 11 \end{bmatrix}$

First column is predictable, so choose $x = (1, 0, 0, 0)^t$.

First four initial columns are linearly independent, first five are clearly linearly dependent.

Relation of linear dependence: $-2A^0 x + 5A^1 x - 3A^2 x - 1A^3 x + A^4 x = 0$.

$p(x) = -2 + 5x - 3x^2 - x^3 + x^4 = (x + 2)(x - 1)^3$
CONCLUSION

“Technology” in teaching (linear algebra) can mean lots of things.

Linear algebra is a perfect discipline for using technology:
- unimaginable dimensions (greater than 4)
- impossible “by-hand” computations (10th power of a $10 \times 10$ matrix)

We owe it to students with applied interests in the modern world to expose them to computation.

Computational power, easily available in our modern world, can illustrate
- applications, such as SVD image compression, and
- theory, such as eigenvalues of companion matrices

Links
- Slides: buzzard.ups.edu/talks.html
- Sage-Enabled Textbook: A First Course in Linear Algebra, linear.ups.edu